

PATHWISE SOLUTIONS TO STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Combining fractional calculus and the Rough Path Theory we study the existence and uniqueness of mild solutions to evolution equations driven by a Hölder continuous function with Hölder exponent in $(1/3, 1/2)$. This theory will be the foundation for establishing that infinite-dimensional white-noise-driven evolution equations with non-trivial diffusion coefficients generate random dynamical systems, a problem which has remained open during the last decades.

1. INTRODUCTION

In the current article, we are interested in developing a pathwise theory for stochastic evolution equations when the noise path is β -Hölder continuous for $\beta \in (1/3, 1/2)$. Our interpretation of pathwise is that for any sufficiently regular noise path we obtain a solution, where the stochastic integral does not produce exceptional sets. In the classical theory of stochastic evolution equation, see for instance the monograph by Da Prato and Zabczyk [7], stochastic integrals are constructed to be a limit in probability of particular random variables defined only almost surely, where the exceptional sets may depend on the initial conditions.

In order to solve this problem, we consider the evolution equations of the form

$$(1.1) \quad \begin{cases} du(t) &= (Au(t) + F(u(t))dt + G(u(t))d\omega(t), \\ u(0) &= u_0 \in V_\delta, \end{cases}$$

in a Hilbert space V , where A is the infinitesimal generator of an analytic semigroup $S(\cdot)$ on V , F and G are nonlinear terms satisfying certain assumptions which will be described in the next sections, ω is a Hölder continuous function with Hölder exponent $\beta \in (1/3, 1/2)$ and $V_\delta = D((-A)^\delta)$ for adequate $\delta > 0$.

As a particular case of driven noises we can consider a fractional Brownian motion (fBm) B^H with Hurst parameter $H \in (1/3, 1/2]$. In general, an fBm B^H with general Hurst parameter $H \in (0, 1)$ is a stochastic process which differs significantly from the standard Brownian motion and, in particular, when $H \neq 1/2$ is not a martingale, so the Ito integrals cannot be used to describe integration when having this type of integrators.

During the last 15 years it can be found in the literature several attempts to develop a theory for stochastic integration for integrators which are not given by a Wiener process, and in particular, for the fractional Brownian motion B^H . One of these attempts is given by the *Rough Path Theory*, and we refer to the monographs by Lyons and Qian [22] and Friz and Victoir [13] for a comprehensive presentation of this theory.

A different technique was developed by Zähle [29], who considered for a fractional Brownian motion with $H > 1/2$ the well-known Young integral. In contrast to the Ito-or Stratonovich integral, that integral can be defined in a pathwise sense. In particular, that integral is given by fractional derivatives, which allow a pathwise estimate of the integrals in terms of integrand and integrator using special norms, see also Nualart and Răşcanu [25]. In this article the authors were able to show the existence and uniqueness of the solution of a finite-dimensional stochastic differential equation driven by a fractional Brownian motion for $H > 1/2$. These results were extended by Maslowski and Nualart [23] to show the existence of mild solutions for stochastic evolution equations. In particular, the mild solution exists for *any* Hölder continuous noise path with Hölder exponent larger than $1/2$ if the coefficients are sufficiently smooth and the linear part of the equation generates an analytic semigroup. Taking advantage of this pathwise interpretation of the stochastic integral, Garrido *et al.* [14] established that, when considering evolution systems driven by fBm with $H > 1/2$, the mild solution generates a random dynamical system.

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To the best of our knowledge there are only a few monographs and articles using the rough path theory to solve stochastic differential equations and stochastic infinite-dimensional evolution equations. In the former set we find the already mentioned books [22], [13], and also Coutin *et al.* [6] and Lejay [21]. In the latter, interesting articles are Gubinelli *et al.* [15], Gubinelli and Tindel [16], Caruana and Friz [3], Caruana *et al.* [4] and Friz and Oberhauser [12]. Our idea to face the study of mild solutions for (1.1) is to combine the rough path theory with classical tools, inspired by the work by Hu and Nualart [17]. In fact, using the ideas in Zähle [29] and Nualart and Răşcanu [25], they have recently extended the pathwise method to the case of dealing with a Hölder continuous driven process with Hölder exponent in $H \in (1/3, 1/2]$. In order to do that, they used a special fractional derivative, the so-called *compensated* fractional derivative. Thanks to that, they were able to formulate an existence and uniqueness result for finite-dimensional stochastic differential equations having coefficients which are sufficiently smooth. We want to stress that, to formulate an operator equation solving this problem, they needed a second equation for the so-called *area* in the space of tensors.

We plan to adapt the techniques in [17] to obtain mild solutions for our infinite-dimensional stochastic evolution equation (1.1), assuming that the linear part generates an analytic semigroup on a separable Hilbert space. However, there are significant differences between both settings: we consider mild solutions for the trajectories part of the evolution equations, which are related to the well known infinite dimensional variation of constants formula. In the existence theorem, we have to use different techniques to obtain a fixed point of the operator equation. As we will see, we have to build the right fixed point argument in order to solve our equation: we will be able to obtain solutions in small intervals that later on can be concatenated to turn out the mild solution over any interval. One has to face with the correct area equation for the tensor mentioned in this infinite-dimensional setting. In a first part of this article we will consider that our evolution equation is driven by a regular noise path, which makes its analysis easier. For this purpose it is of paramount importance to construct a tensor $\omega \otimes_S \omega$ depending on the noise path ω as well as on the semigroup S . Furthermore, this approach has the advantage that the definition of the whole mild solution will become clear, allowing us to understand what is the convenient area equation for our system. The general case requires the additional careful introduction of approximations of the driving noise.

The problem of showing that solutions of stochastic evolution equations generate random dynamical systems is unsolved even for the stochastic partial differential equations driven by the standard Brownian motion. The main difficulties are (i) the stochastic integral is only defined almost surely where the exceptional set may depend on the initial state; and (ii) Kolmogorov's theorem, as cited in Kunita [20] Theorem 1.4.1, is only true for finite dimensional random fields. However, there are some partial results for additive as well as multiplicative noise (see for example, [11], [9, 10], [2] and [24]). In the recent work [14], under appropriate conditions on A , F and G , it has been shown that the stochastic partial differential equation (1.1) above driven by a fBm with Hurst parameter $H \in (1/2, 1)$ generates a random dynamical system.

Thanks to the pathwise results that we are going to establish in this article we can go one step further with respect this unsolved problem, since as an application we are able to derive the existence and uniqueness of mild solutions of infinite-dimensional Brownian-driven evolution equations, with no exceptional sets, which in particular will allow us to prove, in a forthcoming article, that these systems have associated random dynamical systems.

The article is organized as follows. In Section 2 we collect the main tools and give the main assumptions. In particular, we mention important properties of the fractional derivatives. In Section 3 we consider the evolution equation for a *smooth* noise part. That analysis provides us a meaningful definition of solutions of (1.1). We rewrite the equation using fractional derivatives and we then show that we need to present a second equation for the area part. To do this, as we have announced previously, we introduce a tensor defined by the semigroup generated by A , being the crucial step for our considerations, and which constitutes the main difference with respect to the finite-dimensional case developed in [17]. In Section 4 we give the definition of a solution for our system consisting of an path- and an area variables, and an existence and uniqueness theorem is formulated. In the following section we consider the white noise case ($H = 1/2$) and prove how to obtain the tensor defined by the semigroup. We also present two examples to show nonlinearities G that fit the abstract theory.

2. PRELIMINARIES

Let in general V, \tilde{V}, \hat{V} be separable Hilbert spaces. We denote by $L(V, \tilde{V})$ the Banach space of linear operators from V to \tilde{V} and by $L_2(V, \tilde{V}) \subset L(V, \tilde{V})$ the space of Hilbert–Schmidt operators, which is a separable Hilbert space. For $T \in L(\tilde{V}, \hat{V})$, $G \in L_2(V, \tilde{V})$ (and vice versa for $T \in L_2(\tilde{V}, \hat{V})$, $G \in L(V, \tilde{V})$) we have that

$$\|TG\|_{L_2(V, \hat{V})} \leq \|T\|_{L(\tilde{V}, \hat{V})} \|G\|_{L_2(V, \tilde{V})}, \quad \|TG\|_{L_2(V, \hat{V})} \leq \|T\|_{L_2(\tilde{V}, \hat{V})} \|G\|_{L(V, \tilde{V})}.$$

Consider now the separable Hilbert space $(V, |\cdot|, (\cdot, \cdot))$ and assume that $S(\cdot)$ is an analytic semigroup on V generated by an operator A . We also assume that A is a strictly negative operator with a compact inverse, generating a complete orthonormal basis $(e_i)_{i \in \mathbb{N}}$ in V . Let $D((-A)^\delta)$, $\delta \geq 0$, denote the domain of the fractional power $(-A)^\delta$ equipped with the graph norm $|x|_{D((-A)^\delta)} := |(-A)^\delta x|$.

For any $t > 0$ the following inequalities hold

$$(2.1) \quad \|S(t)\|_{L(D((-A)^\delta), D((-A)^\gamma))} = \|(-A)^\gamma S(t)\|_{L(D((-A)^\delta), V)} \leq ct^{\delta-\gamma}, \quad \text{for } \gamma > \delta \geq 0,$$

$$(2.2) \quad \|S(t) - \text{id}\|_{L(D((-A)^\sigma), D((-A)^\theta))} \leq ct^{\sigma-\theta}, \quad \text{for } \sigma - \theta \in [0, 1],$$

since $S(\cdot)$ is an analytic semigroup, see [5], Page 84 (it should be also taken into account that $(-A)^\mu$ is an isomorphism from $D((-A)^{\delta+\mu})$ to $D((-A)^\delta)$ where $\mu \geq 0$).

In addition, the following crucial properties, which proofs are immediate consequences of the previous inequalities, are satisfied for any analytic semigroup $S(\cdot)$:

Lemma 2.1. *For any $0 < \alpha \leq 1$, $0 < \beta \leq 1/2$, $\delta \leq \gamma \in [0, 1)$, there exists a constant $c > 0$ such that for $0 < q < r < s < t$ we have that*

$$\begin{aligned} \|S(t-r) - S(t-q)\|_{L(D((-A)^\delta), D((-A)^\gamma))} &\leq c(r-q)^\alpha (t-r)^{-\alpha-\gamma+\delta}, \\ \|S(t-r) - S(s-r) - S(t-q) + S(s-q)\|_{L(D((-A)^\gamma))} &\leq c(t-s)^\beta (r-q)^{2\beta} (s-r)^{-3\beta}. \end{aligned}$$

In what follows, let us abbreviate $V := V_0$, $V_\delta := D((-A)^\delta)$ with norm $|\cdot|_{V_\delta}$. If $(\lambda_i)_{i \in \mathbb{N}}$ denotes the spectrum of A , since $(e_i)_{i \in \mathbb{N}}$ is an orthonormal basis in V , it follows that $(e_i/\lambda_i^\delta)_{i \in \mathbb{N}}$ is an orthonormal basis in V_δ .

We also need the following estimates concerning the drift G in system (1.1).

Lemma 2.2. *Suppose that $\delta \in [0, 1]$ and $G : V \rightarrow L_2(V, \tilde{V})$ is bounded and twice continuously Fréchet-differentiable with bounded first and second derivatives $DG(u)$ and $D^2G(u)$, for $u \in V$. Let us denote, respectively, by c_G , c_{DG} , c_{D^2G} the bounds for G , DG and D^2G . Then, for $u_1, u_2, v_1, v_2 \in V$, we have*

$$\begin{aligned} \|G(u_1)\|_{L_2(V, \tilde{V})} &\leq c_G, \\ \|G(u_1) - G(v_1)\|_{L_2(V, \tilde{V})} &\leq c_{DG}|u_1 - v_1|, \\ \|DG(u_1) - DG(v_1)\|_{L_2(V \times V, \tilde{V})} &\leq c_{D^2G}|u_1 - v_1|, \\ \|G(u_1) - G(u_2) - DG(u_2)(u_1 - u_2)\|_{L_2(V, \tilde{V})} &\leq c_{D^2G}|u_1 - u_2|^2, \\ \|G(u_1) - G(v_1) - (G(u_2) - G(v_2))\|_{L_2(V, \tilde{V})} \\ &\leq c_{DG}|u_1 - v_1 - (u_2 - v_2)| + c_{D^2G}|u_1 - u_2|(|u_1 - v_1| + |u_2 - v_2|), \\ \|DG(u_1) - DG(v_1) - (DG(u_2) - DG(v_2))\|_{L_2(V \times V, \tilde{V})} \\ &\leq c_{D^2G}|u_1 - v_1 - (u_2 - v_2)| + c_{D^3G}|u_1 - u_2|(|u_1 - v_1| + |u_2 - v_2|). \end{aligned}$$

These estimates follows by the mean value theorem. For the last estimate we refer to Nualart and Răşcanu [25] Lemma 7.1.

Notice that, in particular, $DG : V \rightarrow L_2(V, L_2(V, \tilde{V}))$ (or equivalently, $DG : V \rightarrow L_2(V \times V, \tilde{V})$) is a bilinear map and similarly $D^2G(u)$ is a trilinear map.

Lemma 2.3. *Suppose in addition to the assumptions of Lemma 2.2 that G is three times continuously Fréchet-differentiable where the third derivative is uniformly bounded in $L_2(V, \tilde{V})$. This bound is denoted by c_{D^3G} . Then, for $u_1, u_2, v_1, v_2 \in V$, we have*

$$\begin{aligned} & \|G(u_1) - G(u_2) - DG(u_2)(u_1 - u_2) - (G(v_1) - G(v_2) - DG(v_2)(v_1 - v_2))\|_{L_2(V, \tilde{V})} \\ & \leq c_{D^2G}(|u_1 - u_2| + |v_1 - v_2|)|u_1 - v_1 - (u_2 - v_2)| \\ & \quad + c_{D^3G}|v_1 - v_2||u_2 - v_2|(|u_1 - u_2| + |u_1 - v_1 - (u_2 - v_2)|). \end{aligned}$$

The proof of this lemma can be found in Hu and Nualart [17] Proposition 6.4.

For the nonlinear term F in system (1.1) we assume that it is Lipschitz continuous.

Let us denote by $L_2(V \times V, \tilde{V})$ the space of bilinear continuous mappings B from $V \times V$ which satisfy the Hilbert-Schmidt property

$$\|B\|_{L_2(V \times V, \tilde{V})}^2 := \sum_{i,j=1}^{\infty} |B(e_i, e_j)|_{\tilde{V}}^2 < \infty$$

for some complete orthonormal basis $(e_i)_{i \in \mathbb{N}}$ of V . The topological tensor product of the Hilbert space V is denoted by $V \otimes V$ with norm $\|\cdot\|$. In particular, $V \otimes V$ is a separable Hilbert space. The elements of $V \otimes V$ of the type $v_1 \otimes_V v_2$, with $v_1, v_2 \in V$, are called *elementary tensors or rank one tensors* and for them it holds $\|(v_1, v_2)\| = |v_1||v_2|$. $(e_i \otimes_V e_j)_{i,j \in \mathbb{N}}$ is a complete orthonormal system of $V \otimes V$, where $(e_i)_{i \in \mathbb{N}}$ is the complete orthonormal system for V .

We note that an operator $B \in L_2(V \times V, \tilde{V})$ can be *extended* to an operator $\hat{B} \in L_2(V \otimes V, \tilde{V})$. For weaker conditions we refer to [18] Chapter 2.6. More precisely, there exists a weak Hilbert-Schmidt mapping $p : V \times V \rightarrow V \otimes V$ where $p(e_i, e_j) = e_i \otimes_V e_j$ for $i, j \in \mathbb{N}$. Then \hat{B} on $V \otimes V$ is given by factorization such that $B = \hat{B}p$. In addition, it is easily seen that for the norm of $\hat{B} \in L_2(V \otimes V, \tilde{V})$ it holds

$$\|\hat{B}\|_{L_2(V \otimes V, \tilde{V})}^2 := \sum_{i,j} |\hat{B}(e_i \otimes_V e_j)|_{\tilde{V}}^2 = \sum_{i,j} |B(e_i, e_j)|_{\tilde{V}}^2 = \|B\|_{L_2(V \times V, \tilde{V})}^2.$$

We will write for the extension \hat{B} also the symbol B .

Let $0 \leq T_1 < T_2$. For $\beta \in (0, 1)$ we introduce the space of β -Hölder continuous functions on $[T_1, T_2]$ with values in V denoted by $C_\beta([T_1, T_2]; V)$ with the seminorm

$$\|u\|_\beta = \sup_{s < t \in [T_1, T_2]} \frac{|u(t) - u(s)|}{|t - s|^\beta}.$$

If we consider *all* functions from this linear space having the same value at say T_1 , then $d(u_1, u_2) = \|u_1 - u_2\|_\beta$ creates a complete metric space which will be used later. If we add $|u(T_1)|$ to this seminorm we obtain a Banach space. In particular, this norm is equivalent to the standard norm of Hölder functions on $[T_1, T_2]$.

Let Δ_{T_1, T_2} be the triangle $\{(s, t) : T_1 \leq s < t \leq T_2\}$. We now introduce the space $C_{\beta+\beta'}(\Delta_{T_1, T_2}; V \otimes V)$ of two-forms v with finite norm given by

$$\|v\|_{\beta+\beta'} = \sup_{s < t \in [T_1, T_2]} \frac{\|v(s, t)\|}{|t - s|^{\beta+\beta'}}, \quad \beta + \beta' < 1.$$

Notice that we prefer not to stress the interval $[T_1, T_2]$ in the notation of the previous norms, even though this interval can be different through the text.

Consider $u \in C_\beta([T_1, T_2]; V)$, $\zeta \in C_{\beta'}([T_1, T_2]; V)$ and $v \in C_{\beta+\beta'}(\Delta_{T_1, T_2}; V \otimes V)$ such that the so called *Chen equality* holds, that is, for $T_1 \leq s \leq r \leq t \leq T_2$,

$$(2.3) \quad v(s, r) + v(r, t) + (u(r) - u(s)) \otimes_V (\zeta(t) - \zeta(r)) = v(s, t).$$

Remark 2.4. *An example for v if ζ is continuously differentiable is given by $(u \otimes \zeta)$ where*

$$(2.4) \quad (u \otimes \zeta)(s, t) = \int_s^t (u(\tau) - u(s)) \otimes_V \zeta'(\tau) d\tau = v(s, t).$$

This expression is clearly well-defined and belongs to the space $C_{\beta+\beta'}(\Delta_{T_1, T_2}; V \otimes V)$. Moreover, the Chen equality easily follows.

Now we aim at introducing the so called fractional derivatives and later at giving the pathwise interpretation of the stochastic integral, following the definition in [29].

For $g, \zeta \in C_{\alpha'}([T_1, T_2]; \hat{V})$, being $0 < \alpha < \alpha' < 1$ and \hat{V} some separable Hilbert space which will be given below, we define the fractional derivatives in the Weyl sense by

$$(2.5) \quad \begin{aligned} D_{T_1+}^\alpha g[r] &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{g(r)}{(r-T_1)^\alpha} + \alpha \int_{T_1}^r \frac{g(r)-g(q)}{(r-q)^{1+\alpha}} dq \right) \in \hat{V} \\ D_{T_2-}^\alpha \zeta_{T_2-}[r] &= \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left(\frac{\zeta(r)-\zeta(T_2)}{(T_2-r)^\alpha} + \alpha \int_r^{T_2} \frac{\zeta(r)-\zeta(q)}{(q-r)^{1+\alpha}} dq \right) \in \hat{V}, \end{aligned}$$

where $T_1 \leq r \leq T_2$, and $\zeta_{T_2-}(r) = \zeta(r) - \zeta(T_2)$.

For $v \in C_{\beta+\beta'}(\Delta_{T_1, T_2}; V \otimes V)$ and for $r \in [T_1, T_2]$ we introduce the following fractional derivative

$$(2.6) \quad D_{T_2-}^{1-\alpha} v[r] = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{v(r, T_2)}{(T_2-r)^{1-\alpha}} + (1-\alpha) \int_r^{T_2} \frac{v(r, \tau)}{(\tau-r)^{2-\alpha}} d\tau \right).$$

The following formulas play an essential role in our results (see [29]):

$$(2.7) \quad (-1)^\alpha \int_{T_1}^{T_2} D_{T_1+}^\alpha g[r] \zeta(r) dr = \int_{T_1}^{T_2} g(r) D_{T_2-}^\alpha \zeta[r] dr,$$

for $g \in I_{T_1+}^\alpha(L^p((T_1, T_2); \mathbb{R}))$, $\zeta \in I_{T_2-}^\alpha(L^q((T_1, T_2); \mathbb{R}))$ with $1/p + 1/q \leq 1$ (for the definition of the spaces $I_{T_1+}^\alpha(L^p((T_1, T_2); \mathbb{R}))$ and $I_{T_2-}^\alpha(L^q((T_1, T_2); \mathbb{R}))$ we refer, for instance, to Samko *et al.* [27]). If now we assume that $g(T_1+)$, $\zeta(T_1+)$, $\zeta(T_2-)$ exist, being respectively the right side limit of g at T_1 and the right and left side limits of ζ at T_1 , T_2 , and that $g_{T_1+} \in I_{T_1+}^\alpha(L^p((T_1, T_2); \mathbb{R}))$, $\zeta_{T_2-} \in I_{T_2-}^\alpha(L^q((T_1, T_2); \mathbb{R}))$ with $1/p + 1/q \leq 1$, then

$$(2.8) \quad \int_{T_1}^{T_2} g d\zeta = (-1)^\alpha \int_{T_1}^{T_2} D_{T_1+}^\alpha g_{T_1+}[r] D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r] dr + g(T_1+)(\zeta(T_2-) - \zeta(T_1+)).$$

Here $g_{T_1+}(\cdot) = g(\cdot) - g(T_1+)$ and $\zeta_{T_2-}(\cdot) = \zeta(\cdot) - \zeta(T_2-)$. In addition, when $\alpha p < 1$ and $g(T_1+)$ exists and $g \in I_{T_1+}^\alpha(L^p((T_1, T_2); \mathbb{R}))$, (2.8) can be rewritten as

$$(2.9) \quad \int_{T_1}^{T_2} g d\zeta = (-1)^\alpha \int_{T_1}^{T_2} D_{T_1+}^\alpha g[r] D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r] dr.$$

Notice that in the case that ζ is not Lipschitz continuous we cannot define the integral on the left hand side of (2.8) in the classical way. Nevertheless, when g and ζ are Hölder continuous with exponents γ, β resp., and $\alpha < \gamma$, $1 - \alpha < \beta$, we can do it. In particular, assume that $g \in C_\gamma([T_1, T_2]; L_2(V, \tilde{V}))$, $\zeta \in C_\beta([T_1, T_2]; V)$ for $0 < \alpha < \gamma$, $1 - \alpha < \beta$. Let us define

$$(2.10) \quad \int_{T_1}^{T_2} g(r) d\zeta(r) = (-1)^\alpha \int_{T_1}^{T_2} D_{T_1+}^\alpha g[r] D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r] dr.$$

This expression can be also interpreted as a fractional integration by parts formula. By the separability of \tilde{V} , Pettis' theorem and by

$$\begin{aligned}
 (2.11) \quad & \left| \int_{T_1}^{T_2} D_{T_1+}^\alpha g[r] D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r] dr \right|_{\tilde{V}} \leq \int_{T_1}^{T_2} |D_{T_1+}^\alpha g[r] D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r]|_{\tilde{V}} dr \\
 & \leq \int_{T_1}^{T_2} \|D_{T_1+}^\alpha g[r]\|_{L_2(V, \tilde{V})} |D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r]| dr \\
 & \leq c \|\zeta\|_\beta \int_{T_1}^{T_2} (\|g(T_1)\|_{L_2(V, \tilde{V})} (r - T_1)^{-\alpha} (T_2 - r)^{\alpha+\beta-1} + \|g\|_\gamma (r - T_1)^{\gamma-\alpha} (T_2 - r)^{\alpha+\beta-1}) dr \\
 & \leq c \|\zeta\|_\beta \left(\|g(T_1)\|_{L_2(V, \tilde{V})} (T_2 - T_1)^\beta + c \|g\|_\gamma (T_2 - T_1)^{\beta+\gamma} \right)
 \end{aligned}$$

this integral is well defined. Indeed,

$$(2.12) \quad \|D_{T_1+}^\alpha g[r]\|_{L_2(V, \tilde{V})} \leq c \left(\frac{\|g(T_1)\|_{L_2(V, \tilde{V})}}{(r - T_1)^\alpha} + \frac{\|g\|_\gamma}{(r - T_1)^{\alpha-\gamma}} \right),$$

and, if $1-\alpha < \beta$, the expression $D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r]$ is well defined, actually, it is simple to obtain that $D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r] \leq \|\zeta\|_\beta (T_2 - r)^{\beta+\alpha-1}$.

Let $\{\tilde{e}_i\}_{i \in \mathbb{N}}$ a complete orthonormal basis of \tilde{V} . Denote by π_m and $\tilde{\pi}_m$ the orthogonal projections on $\{e_1, \dots, e_m\}$ and $\{\tilde{e}_1, \dots, \tilde{e}_m\}$, respectively, and define $g_j = (\tilde{\pi}_j - \tilde{\pi}_{j-1})g$, $\zeta_i = (\pi_i - \pi_{i-1})\zeta$ and $g_{ji} = (\tilde{\pi}_j - \tilde{\pi}_{j-1})g(\pi_i - \pi_{i-1})$. The above estimates allow to exchange the sum and the integral such that we have

$$(2.13) \quad \int_{T_1}^{T_2} g(r) d\zeta(r) = \sum_j \left(\sum_i \int_{T_1}^{T_2} D_{T_1+}^\alpha g_{ji}[r] D_{T_2-}^{1-\alpha} \zeta_{i T_2-}[r] dr \right) \tilde{e}_j$$

and

$$\left| \int_{T_1}^{T_2} g(r) d\zeta(r) \right|_{\tilde{V}} = \left(\sum_j \left| \sum_i \int_{T_1}^{T_2} g(r)_{ji} d\zeta_i(r) \right|^2 \right)^{\frac{1}{2}} < \infty$$

where we have used that

$$\begin{aligned}
 \|D_{T_1+}^\alpha g[r]\|_{L_2(V, \tilde{V})} &= \left(\sum_{ij} |D_{T_1+}^\alpha g(\cdot)_{ji}[r]|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{ij} \left(\frac{1}{\Gamma(1-\alpha)} \left(\frac{g_{ji}(r)}{(r - T_1)^\alpha} + \alpha \int_{T_1}^r \frac{g_{ji}(r) - g_{ji}(q)}{(r - q)^{1+\alpha}} dq \right) \right)^2 \right)^{\frac{1}{2}} \\
 &\leq \sqrt{2}c \left(\frac{(\sum_{ij} g_{ji}(r)^2)^{\frac{1}{2}}}{(r - T_1)^\alpha} + \left(\sum_{ij} \left(\int_{T_1}^r \frac{g_{ji}(r) - g_{ji}(q)}{(r - q)^{1+\alpha}} dq \right)^2 \right)^{\frac{1}{2}} \right) \\
 &\leq \sqrt{2}c \left(\frac{\|g(r)\|_{L_2(V, \tilde{V})}}{(r - T_1)^\alpha} + \int_{T_1}^r \frac{\|g(r) - g(q)\|_{L_2(V, \tilde{V})}}{(r - q)^{1+\alpha}} dq \right) \\
 &\leq c(r - T_1)^{-\alpha} (1 + \|g\|_\gamma (1 + (r - T_1)^\gamma)).
 \end{aligned}$$

In a similar manner we can also define integrals with values in the separable Hilbert space $V \otimes V$ when $g(r) \in L_2(V, \mathbb{R}) \cong V$ by

$$\int_{T_1}^{T_2} g(r) \otimes_V d\zeta(r).$$

Remark 2.5. Suppose that g_{ji} satisfies the assumptions for (2.9) but is not Hölder continuous in general. In addition suppose that $r \mapsto \|D_{T_1+}^\alpha g[r]\|_{L_2(V, \tilde{V})} |D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r]|$ is integrable. Then we can define the integrals (2.10) and (2.13). This will be used later on for $g(r) = S(T_2 - r)f(r)$ where f is an appropriate function. Note that $r \mapsto S(T_2 - r)x$, $x \in V$ is not Hölder continuous.

Suppose now that $g(r) = G(u(r))$ where $u \in C_\gamma([T_1, T_2]; V)$ for $\alpha < \gamma$, $\alpha + \gamma > 1$ with G having a bounded Fréchet derivative DG . Then

$$(2.14) \quad \int_{T_1}^{T_2} G(u) d\zeta = (-1)^\alpha \int_{T_1}^{T_2} D_{T_1+}^\alpha G(u(\cdot))[r] D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r] dr$$

is well defined because $G(u(\cdot))$ is γ -Hölder continuous. Assuming in addition that G has a second bounded derivative we can rewrite the integral in (2.14) as follows

$$\begin{aligned} \int_{T_1}^{T_2} G(u) d\zeta &= (-1)^\alpha \int_{T_1}^{T_2} D_{T_1+}^\alpha (G(u(\cdot)) - DG(u(\cdot))(u - u(T_1), \cdot))[r] D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r] dr \\ &\quad + (-1)^\alpha \int_{T_1}^{T_2} D_{T_1+}^\alpha DG(u(\cdot))(u - u(T_1), \cdot)[r] D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r] dr. \end{aligned}$$

Suppose now that the above condition $\gamma > \alpha$ is not satisfied. Then $D_{T_1+}^\alpha G(u)$ is not well defined, in general. In this case it has sense to rewrite (2.14) by using the so called *compensated fractional derivative*

$$(2.15) \quad \begin{aligned} \hat{D}_{T_1+}^\alpha G(u(\cdot))[r] &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{G(u(r))}{(r-T_1)^\alpha} \right. \\ &\quad \left. + \alpha \int_{T_1}^r \frac{G(u(r)) - G(u(q)) - DG(u(q))(u(r) - u(q), \cdot)}{(r-q)^{\alpha+1}} dq \right) \in L_2(V, \tilde{V}) \end{aligned}$$

if $2\gamma > \alpha$. By making some computations, it is not difficult to see that there is the following relation between the fractional derivative and compensated one:

$$(2.16) \quad D_{T_1+}^\alpha (G(u(\cdot)) - DG(u(\cdot))(u(\cdot) - u(T_1), \cdot))[r] = \hat{D}_{T_1+}^\alpha G(u(\cdot))[r] - D_{T_1+}^\alpha DG(u(\cdot))[r](u(r) - u(T_1), \cdot).$$

In addition, we have

$$(2.17) \quad D_{T_2-}^{1-\alpha} v(T_1, \cdot)_{T_2-}[r] = -D_{T_2-}^{1-\alpha} v[r] + (u(r) - u(T_1)) \otimes_V D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r],$$

where u, ζ, v are coupled by the Chen equality (2.3). This then enables us to express (2.14) under a weaker regularity condition (if $\gamma > \alpha$ is not satisfied, but for instance $2\gamma > \alpha$) in the way

$$(2.18) \quad \begin{aligned} \int_{T_1}^{T_2} G(u) d\zeta &= (-1)^\alpha \int_{T_1}^{T_2} \hat{D}_{T_1+}^\alpha G(u(\cdot))[r] D_{T_2-}^{1-\alpha} \zeta_{T_2-}[r] dr \\ &\quad - (-1)^{2\alpha-1} \int_{T_1}^{T_2} D_{T_1+}^{2\alpha-1} DG(u(\cdot))[r] D_{T_2-}^{1-\alpha} \mathcal{D}_{T_2-}^{1-\alpha} v[r] dr. \end{aligned}$$

If $\hat{D}_{T_1+}^\alpha G(u(\cdot))[r]$ has now the right regularity then we can define the first integral on the right hand side of the last formula similar to (2.10).

Now let us consider an integral having the structure of the second integral, namely

$$(2.19) \quad \int_{T_1}^{T_2} D_{T_1+}^{2\alpha-1} g(\cdot)[r] D_{T_2-}^{1-\alpha} \mathcal{D}_{T_2-}^{1-\alpha} v[r] dr$$

for some $v \in C_{\beta+\beta'}(\Delta_{T_1, T_2}; V \otimes V)$. Then it holds

$$(2.20) \quad |D_{T_2-}^{1-\alpha} \mathcal{D}_{T_2-}^{1-\alpha} v[r]| \leq c \|v\|_{\beta+\beta'} (T_2 - r)^{\beta+\beta'+2\alpha-2}$$

and hence this integral can be defined in a similar manner than (2.10) provided that $g(\cdot)$ is γ -Hölder continuous with $0 < \alpha < 1$, $\gamma + 1 > 2\alpha$, and, in particular, with $\beta' > \beta > 1 - \alpha$. For details in finite dimension we refer to [17].

In the next sections we will give sense to the integrals appearing in the definition of mild solution to the infinite dimensional equation (1.1).

3. FORMULATION OF (1.1) FOR SMOOTH PATHS ω

Throughout this section, we assume that the driven path $\omega : [0, T] \rightarrow V$ in the system (1.1) is smooth in the sense that ω is continuous at any t and continuously differentiable except at finitely many points. Then we derive a system of equations which is needed to define a solution when the noise is only Hölder continuous, the case to be considered in the next section. When the path ω is only V -valued Hölder continuous with Hölder exponent in $(1/3, 1/2)$, we will consider a piecewise linear approximation ω^n of ω , for which we can apply the results that we are going to establish throughout this section.

In what follows we assume that $\tilde{V} = V_\delta$ for $\delta \in [0, 1]$, and this in particular means that in the latter space we identify $(\tilde{e}_i)_{i \in \mathbb{N}}$ with $(e_i/\lambda_i^\delta)_{i \in \mathbb{N}}$. Under such a choice, we consider $G : V \rightarrow L_2(V, V_\delta)$.

For the fixed regular ω , we study the equation

$$(3.1) \quad u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds + \int_0^t S(t-s)G(u(s))d\omega(s).$$

Lemma 3.1. ([26]) *Under the condition that F, G are Lipschitz continuous, the above equation has a unique global solution which depends continuously on $u_0 \in V_\delta$. Moreover, $u \in C_\beta([0, T]; V)$ for $\beta \leq \delta \in [0, 1]$.*

If in addition we assume that G is twice continuously Fréchet-differentiable with bounded second derivative, following the steps in the proof of Theorem 3.3 in [17], we can rewrite the above equation as

$$(3.2) \quad \begin{aligned} u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds + (-1)^\alpha \int_0^t \hat{D}_{0+}^\alpha (S(t-\cdot)G(u(\cdot))) [r] D_{t-}^{1-\alpha} \omega_{t-} [r] dr \\ - (-1)^{2\alpha-1} \int_0^t D_{0+}^{2\alpha-1} (S(t-\cdot)DG(u(\cdot))) [r] D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} (u \otimes \omega) [r] dr, \end{aligned}$$

where the compensated fractional derivative is defined as in (2.15) and the fractional derivative of $(u \otimes \omega)$ is defined as in (2.6).

We want to point out that, when the noise is not regular, as we will show in the following section, in order to give a meaningful definition of the solution of (1.1), we need an equation to determine the corresponding counterpart of $(u \otimes \omega)$. For this reason, now we aim at getting such expression in the case that ω is smooth. Firstly, choosing $\zeta = \omega$ in (2.4), we can express the tensor as follows

$$(3.3) \quad \begin{aligned} (u \otimes \omega)(s, t) = \int_s^t (S(\xi - s) - \text{id})u(s) \otimes_V \omega'(\xi) d\xi + \int_s^t \int_s^\xi S(\xi - r)F(u(r))dr \otimes_V \omega'(\xi) d\xi \\ + \int_s^t \int_s^\xi S(\xi - r)G(u(r))\omega'(r)dr \otimes_V \omega'(\xi) d\xi. \end{aligned}$$

Fix $\delta \in [0, 1]$ and let $\beta' \in (1/3, 1/2)$. For $\alpha \in (0, 1)$, $0 \leq s \leq t \leq T$ and the semigroup S introduced in Section 2, we consider the mapping $\omega_S : \Delta_{0,T} \rightarrow L_2(V_\delta; V \otimes V)$ defined by the following integral

$$\omega_S(s, t) \cdot = (-1)^{-\alpha} \int_s^t (S(\xi - s) \cdot) \otimes_V \omega'(\xi) d\xi.$$

It can be also easily seen that for any $e \in V$ there exists $c \geq 0$ such that

$$\|\omega_S(s, t)e - \omega_S(r, t)e\| \leq c(r-s)^{\beta'}|e|, \quad 0 \leq s \leq r \leq t.$$

We consider $(\omega \otimes_S \omega) : \Delta_{0,T} \times L_2(V, V_\delta) \rightarrow V \otimes V$ given by

$$(3.4) \quad \begin{aligned} E(\omega \otimes_S \omega)(s, t) &= \int_s^t \int_s^\xi S(\xi - r)E\omega'(r)dr \otimes_V \omega'(\xi) d\xi \\ &= \int_s^t \int_r^t S(\xi - r)E\omega'(r) \otimes_V \omega'(\xi) d\xi dr = (-1)^\alpha \int_s^t \omega_S(r, t)E(\omega'(r))dr. \end{aligned}$$

Let us recall that $L_2(V, V_\delta)$ is a separable Hilbert space with an orthonormal basis $(E_{ij})_{i,j \in \mathbb{N}}$ derived from the basis $(e_k)_{k \in \mathbb{N}}$ of V and $(e_k/\lambda_k^\delta)_{k \in \mathbb{N}}$ of V_δ as follows:

$$E_{ij}e_k = \begin{cases} 0 & : j \neq k \\ \frac{e_i}{\lambda_i^\delta} & : j = k. \end{cases}$$

Suppose that

$$(3.5) \quad \sum_{i=1}^{\infty} \lambda_i^{-1-2\delta} < \infty.$$

Then for the smooth path ω we have

$$\begin{aligned} \|E_{ij}(\omega \otimes_S \omega)(s, t)\|^2 &= \left\| \int_s^t \int_r^t S(\xi - r) E_{ij} \omega'(r) \otimes_V \omega'(\xi) d\xi dr \right\|^2 \\ &= \left\| \int_s^t \int_r^t e^{-\lambda_i(\xi-r)} \frac{e_i}{\lambda_i^\delta} (\omega'(r), e_j) \otimes_V \omega'(\xi) d\xi dr \right\|^2, \\ \sum_{ij} \|E_{ij}(\omega \otimes_S \omega)(s, t)\|^2 &\leq \left(\int_s^t |\omega'(\xi)|^2 d\xi \right)^2 \sum_i \frac{1}{2\lambda_i^{1+2\delta}} \int_s^t (1 - e^{-2\lambda_i(t-r)}) dr < \infty. \end{aligned}$$

In particular, since ω is smooth we can conclude that $(\omega \otimes_S \omega) \in C_{2\beta'}(\Delta_{0,T}; L_2(L_2(V, V_\delta), V \otimes V))$. Indeed, from the above inequality, there exists c such that

$$\sum_{ij} \|E_{ij}(\omega \otimes_S \omega)(s, t)\|^2 \leq c(t-s)^2.$$

The following equality is interpreted to be the Chen equality for $(\omega \otimes_S \omega)$: for $0 \leq s \leq r \leq t \leq T$ it holds

$$(3.6) \quad \begin{aligned} &E(\omega \otimes_S \omega)(s, r) + E(\omega \otimes_S \omega)(r, t) \\ &+ \int_r^t S(\xi - r) \int_s^r S(r - q) E\omega'(q) dq \otimes_V \omega'(\xi) d\xi = E(\omega \otimes_S \omega)(s, t). \end{aligned}$$

Taking into account (3.4), the last integral on the right hand side of (3.3) can be written as

$$\begin{aligned} & - \int_s^t G(u(r)) D_1(\omega \otimes_S \omega)(r, t) dr \\ &= - \int_s^t (G(u(r)) - DG(u(r))(u(r) - u(s), \cdot)) D_1(\omega \otimes_S \omega)(r, t) dr \\ & - \int_s^t DG(u(r))(u(r) - u(s), \cdot) D_1(\omega \otimes_S \omega)(r, t) dr \\ (3.7) \quad &= -(-1)^\alpha \int_s^t \hat{D}_{s+}^\alpha G(u(\cdot))[r] D_{t-}^{1-\alpha}(\omega \otimes_S \omega)(\cdot, t)_{t-}[r] dr \\ &+ (-1)^\alpha \int_s^t D_{s+}^\alpha DG(u(\cdot))(u(r) - u(s), \cdot)[r] D_{t-}^{1-\alpha}(\omega \otimes_S \omega)(\cdot, t)_{t-}[r] dr \\ &- (-1)^\alpha \int_s^t D_{s+}^\alpha DG(u(\cdot))[r] D_{t-}^{1-\alpha}(u \otimes (\omega \otimes_S \omega)(\cdot, t))(s, \cdot)_{t-}[r] dr. \end{aligned}$$

Notice that in the previous expression we have used (2.10) and (2.16). Moreover, for $\tilde{E} \in L_2(V \times V, V_\delta)$, we set

$$\begin{aligned} \tilde{E}(u \otimes (\omega \otimes_S \omega))(\cdot, t)(s, r) &= \int_s^r \tilde{E}(u(q) - u(s), \cdot) D_1(\omega \otimes_S \omega)(q, t) dq \\ &= - \int_s^r \int_q^t S(\xi - q) \tilde{E}(u(q) - u(s), \omega'(q)) \otimes_V \omega'(\xi) d\xi dq \in V \otimes V \end{aligned}$$

such that

$$\tilde{E} D_2(u \otimes (\omega \otimes_S \omega))(\cdot, t)(s, \cdot)[r] = \tilde{E}(u(r) - u(s), \cdot) D_1(\omega \otimes_S \omega)(r, t),$$

which gives us the last integral on the right hand side of (3.7). From now on we write $(u \otimes (\omega \otimes_S \omega))(t)$ instead of $(u \otimes (\omega \otimes_S \omega))(\cdot, t)$.

We consider the separable Hilbert space $L_2(V \times V, V_\delta)$ equipped with the complete orthonormal basis $(\tilde{E}_{ijk})_{i,j,k \in \mathbb{N}}$

$$\tilde{E}_{ijk}(e_l, e_m) = \begin{cases} 0 & : j \neq l \text{ or } k \neq m \\ \frac{e_i}{\lambda_i^\delta} & : j = l \text{ and } k = m. \end{cases}$$

Under the assumption (3.5), similarly to the above estimate for $(\omega \otimes_S \omega)$, we have that

$$\begin{aligned} & \sum_{ijk} \left\| \int_s^r \int_q^t S(\xi - q) \tilde{E}_{ijk}(u(q) - u(s), \omega'(q)) \otimes_V \omega'(\xi) d\xi dq \right\|^2 \\ & \leq \sum_i \frac{1}{2\lambda_i^{1+2\delta}} \int_s^t (1 - e^{-2\lambda_i(t-q)}) dq \int_s^t |\omega'(\xi)|^2 d\xi \int_s^t |u(q) - u(s)|^2 |\omega'(q)|^2 dq < \infty, \end{aligned}$$

which shows in particular that

$$(u \otimes (\omega \otimes_S \omega)) \in C_{\beta+\beta'}(\Delta_{0,T}; L_2(L_2(V \times V, V_\delta), V \otimes V)).$$

Lemma 3.2. *Suppose that (3.5) holds. For $0 \leq s \leq t \leq T$, $(u \otimes \omega)$ satisfies the equation*

$$\begin{aligned} (3.8) \quad (u \otimes \omega)(s, t) &= \int_s^t (S(\xi - s) - \text{id})u(s) \otimes_V \omega'(\xi) d\xi + \int_s^t \int_s^\xi S(\xi - r) F(u(r)) dr \otimes_V \omega'(\xi) d\xi \\ &- (-1)^\alpha \int_s^t \hat{D}_{s+}^\alpha G(u(\cdot))[r] D_{t-}^{1-\alpha} (\omega \otimes_S \omega)(\cdot, t)_{t-}[r] dr \\ &+ (-1)^{2\alpha-1} \int_s^t D_{s+}^{2\alpha-1} DG(u(\cdot))[r] D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} (u \otimes (\omega \otimes_S \omega)(t))[r] dr. \end{aligned}$$

Proof. Let us consider $\tilde{E} \in L_2(V \times V, V_\delta)$. Since, for $0 \leq s \leq r \leq q \leq t \leq T$,

$$\begin{aligned} & - \int_s^r \int_\tau^t S(\xi - \tau) \tilde{E}(u(\tau) - u(s), \omega'(\tau)) \otimes_V \omega'(\xi) d\xi d\tau \\ & - \int_r^q \int_\tau^t S(\xi - \tau) \tilde{E}(u(\tau) - u(r), \omega'(\tau)) \otimes_V \omega'(\xi) d\xi d\tau \\ & - \int_r^q \int_\tau^q S(\xi - \tau) \tilde{E}(u(r) - u(s), \omega'(\tau)) \otimes_V \omega'(\xi) d\xi d\tau \\ & - \int_r^q \int_q^t S(\xi - \tau) \tilde{E}(u(r) - u(s), \omega'(\tau)) \otimes_V \omega'(\xi) d\xi d\tau \\ & = - \int_s^q \int_\tau^t S(\xi - \tau) \tilde{E}(u(\tau) - u(s), \omega'(\tau)) \otimes_V \omega'(\xi) d\xi d\tau \end{aligned}$$

we have

$$\begin{aligned} (3.9) \quad & \tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, r) + \tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(r, q) - \tilde{E}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(r, q) \\ & = \tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, q) + \int_r^q \int_q^t S(\xi - \tau) \tilde{E}(u(r) - u(s), \omega'(\tau)) \otimes_V \omega'(\xi) d\xi d\tau. \end{aligned}$$

In particular, when $q = t$, we have

$$\begin{aligned} (3.10) \quad & \tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, r) + \tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(r, t) - \tilde{E}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(r, t) \\ & = \tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, t). \end{aligned}$$

Since $(\omega \otimes_S \omega)(t, t) = 0$ the above expression is exactly the Chen equality for $(u \otimes (\omega \otimes_S \omega))(t)$:

$$\begin{aligned} & \tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, r) + \tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(r, t) \\ & + \tilde{E}(u(r) - u(s), \cdot)((\omega \otimes_S \omega)(t, t) - (\omega \otimes_S \omega)(r, t)) = \tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, t). \end{aligned}$$

Thanks to (3.9) we have

$$\begin{aligned}
& \tilde{E}D_{t-}^{1-\alpha}(u \otimes (\omega \otimes_S \omega)(t))(s, \cdot)_{t-}[r] \\
&= \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{\tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, r) - \tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, t)}{(t-r)^{1-\alpha}} \right. \\
&+ (1-\alpha) \int_r^t \frac{\tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, r) - \tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, \theta)}{(\theta-r)^{2-\alpha}} d\theta \Big) \\
&= \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{-\tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(r, t) + \tilde{E}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(r, t)}{(t-r)^{1-\alpha}} \right. \\
&+ (1-\alpha) \int_r^t \frac{-\tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(r, \theta) + \tilde{E}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(r, \theta)}{(\theta-r)^{2-\alpha}} d\theta \\
&+ (1-\alpha) \int_r^t \frac{\int_r^\theta \int_\theta^t S(\xi - \tau) \tilde{E}(u(r) - u(s), \omega'(\tau)) \otimes_V \omega'(\xi) d\xi d\tau}{(\theta-r)^{2-\alpha}} d\theta \Big) \\
&= -\tilde{E}\mathcal{D}_{t-}^{1-\alpha}(u \otimes (\omega \otimes_S \omega)(t))[r] \\
&+ \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{\tilde{E}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(r, t)}{(t-r)^{1-\alpha}} + (1-\alpha) \int_r^t \frac{\tilde{E}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(r, \theta)}{(\theta-r)^{2-\alpha}} d\theta \right. \\
&+ (1-\alpha) \int_r^t \frac{\int_r^\theta \int_\theta^t S(\xi - \tau) \tilde{E}(u(r) - u(s), \omega'(\tau)) \otimes_V \omega'(\xi) d\xi d\tau}{(\theta-r)^{2-\alpha}} d\theta \Big).
\end{aligned}$$

Furthermore, by (3.6) we have

$$\begin{aligned}
& \tilde{E}D_{t-}^{1-\alpha}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(\cdot, t)_{t-}[r] = \tilde{E}(u(r) - u(s), \cdot)D_{t-}^{1-\alpha}(\omega \otimes_S \omega)(\cdot, t)_{t-}[r] \\
&= \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{\tilde{E}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(r, t)}{(t-r)^{1-\alpha}} \right. \\
&+ (1-\alpha) \int_r^t \frac{\tilde{E}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(r, t) - \tilde{E}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(\theta, t)}{(\theta-r)^{2-\alpha}} d\theta \Big) \\
&= \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{\tilde{E}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(r, t)}{(t-r)^{1-\alpha}} \right. \\
&+ (1-\alpha) \int_r^t \frac{\tilde{E}(u(r) - u(s), \cdot)(\omega \otimes_S \omega)(r, \theta)}{(\theta-r)^{2-\alpha}} d\theta \\
&+ (1-\alpha) \int_r^t \frac{\int_r^\theta \int_\theta^t S(\xi - \tau) \tilde{E}(u(r) - u(s), \omega'(\tau)) \otimes_V \omega'(\xi) d\xi d\tau}{(\theta-r)^{2-\alpha}} d\theta \Big).
\end{aligned}$$

Plugging the above expression into the previous expression of $\tilde{E}D_{t-}^{1-\alpha}(u \otimes (\omega \otimes_S \omega)(t))(s, \cdot)_{t-}[r]$, we obtain

$$\begin{aligned}
& \tilde{E}D_{t-}^{1-\alpha}(u \otimes (\omega \otimes_S \omega)(t))(s, \cdot)_{t-}[r] \\
&= -\tilde{E}\mathcal{D}_{t-}^{1-\alpha}(u \otimes (\omega \otimes_S \omega)(t))[r] + \tilde{E}(u(r) - u(s), \cdot)D_{t-}^{1-\alpha}(\omega \otimes_S \omega)(\cdot, t)_{t-}[r].
\end{aligned}$$

Note that the previous equality shows a similar connection between the fractional derivative and the compensated fractional derivative obtained previously in (2.17).

Hence, using (3.7) and the fractional integration by parts formula (2.7), we obtain that $(u \otimes \omega)$ satisfies the equation

$$\begin{aligned}
(u \otimes \omega)(s, t) &= \int_s^t (S(\xi - s) - \text{id})u(s) \otimes_V \omega'(\xi) d\xi + \int_s^t \int_s^\xi S(\xi - r)F(u(r))dr \otimes_V \omega'(\xi) d\xi \\
&\quad - (-1)^\alpha \int_s^t \hat{D}_{s+}^\alpha G(u(\cdot))[r] D_{t-}^{1-\alpha}(\omega \otimes_S \omega)(\cdot, t)_{t-}[r] dr \\
&\quad + (-1)^\alpha \int_s^t D_{s+}^\alpha DG(u(\cdot))[r] \mathcal{D}_{t-}^{1-\alpha}(u \otimes (\omega \otimes_S \omega)(t))[r] dr \\
&= \int_s^t (S(\xi - s) - \text{id})u(s) \otimes_V \omega'(\xi) d\xi + \int_s^t \int_s^\xi S(\xi - r)F(u(r))dr \otimes_V \omega'(\xi) d\xi \\
&\quad - (-1)^\alpha \int_s^t \hat{D}_{s+}^\alpha G(u(\cdot))[r] D_{t-}^{1-\alpha}(\omega \otimes_S \omega)(\cdot, t)_{t-}[r] dr \\
&\quad + (-1)^{2\alpha-1} \int_s^t D_{s+}^{2\alpha-1} DG(u(\cdot))[r] D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha}(u \otimes (\omega \otimes_S \omega)(t))[r] dr
\end{aligned}$$

which completes the proof. \square

We note that the last two integrals in (3.8) are well defined. In particular, we apply the operator $D_{t-}^{1-\alpha}(\omega \otimes_S \omega)(\cdot, t)_{t-}[r]$, which has values in the separable Hilbert space $L_2(L_2(V, V_\delta), V \otimes V)$, to $\hat{D}_{s+}^\alpha G(u(\cdot))[r]$, which is contained in the separable Hilbert space $L_2(V, V_\delta)$. Then we can use the definition of a Hilbert space valued integral of Section 2. Similar we can argue for the last integral of (3.8). In the Appendix, we will prove that $(u \otimes \omega)$ given by (3.8) satisfies the Chen equality (see Lemma 6.5).

Let us now deal with the structure of $(u \otimes (\omega \otimes_S \omega))$.

Lemma 3.3. *Suppose (3.5) holds. Let $\tilde{E} \in L_2(V \otimes V, V_\delta)$. Then for $0 \leq s \leq q \leq t \leq T$ the expression $\tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, q)$ satisfies the equation*

$$\begin{aligned}
(3.11) \quad \tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, q) &= -(-1)^\alpha \int_s^q \omega_S(r, t) \tilde{E}(u(r) - u(s), \omega'(r)) dr \\
&= - \int_s^q \hat{D}_{s+}^\alpha \omega_S(\cdot, t) \tilde{E}(u(\cdot) - u(s), \cdot)[r] D_{q-}^{1-\alpha} \omega_{q-}[r] dr \\
&\quad + (-1)^{\alpha-1} \int_s^q D_{s+}^{2\alpha-1} \tilde{E}(u(\cdot) - u(s), \cdot)[r] D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha}(\omega_S(t) \otimes \omega)[r] dr \\
&\quad + (-1)^{\alpha-1} \int_s^q D_{s+}^{2\alpha-1} \omega_S(\cdot, t)[r] \tilde{E} D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha}(u \otimes \omega)(t)[r] dr,
\end{aligned}$$

where $(\omega_S(t) \otimes \omega)$ is defined for $s \leq \tau \leq t$, $E \in L_2(V, V_\delta)$ as

$$\begin{aligned}
E(\omega_S(t) \otimes \omega)(s, \tau) &= \int_s^\tau (\omega_S(r, t) - \omega_S(s, t)) E d\omega(r) \\
&= \int_s^\tau (\omega_S(r, t) - \omega_S(r, \tau)) E d\omega(r) + \int_s^\tau \omega_S(r, \tau) E d\omega(r) - \int_s^\tau \omega_S(s, t) E d\omega(r) \\
&= \omega_S(\tau, t) \int_s^\tau S(\tau - r) E d\omega(r) + (-1)^{-\alpha} E(\omega \otimes_S \omega)(s, \tau) - \omega_S(s, t) E(\omega(\tau) - \omega(s)).
\end{aligned}$$

We note that for smooth ω the term $E(\omega_S(t) \otimes \omega)(s, \tau)$ is defined in the sense of (2.4).

Proof. We define

$$f_{\tilde{E}} : L_2(V_\delta, V \otimes V) \times V \rightarrow L_2(V, V \otimes V) : f_{\tilde{E}}(Q, u) = Q(\tilde{E}(u, \cdot)).$$

From (3.4) we have for smooth ω that

$$\begin{aligned}\tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, q) &= -(-1)^\alpha \int_s^q \omega_S(r, t) \tilde{E}(u(r) - u(s), \omega'(r)) dr \\ &= -(-1)^\alpha \int_s^q f_{\tilde{E}}(\omega_S(r, t), u(r) - u(s)) \omega'(r) dr.\end{aligned}$$

Following (2.16) or Theorem 3.3 in [17], we have

$$\begin{aligned}(3.12) \quad & \int_s^q f_{\tilde{E}}(\omega_S(r, t), u(r) - u(s)) \omega'(r) dr \\ &= (-1)^\alpha \int_s^q \hat{D}_{s+}^\alpha f_{\tilde{E}}(\omega_S(\cdot, t), u(\cdot) - u(s)) [r] D_{q-}^{1-\alpha} \omega_{q-} [r] dr \\ &\quad - (-1)^\alpha \int_s^q (\omega_S(r, t) - \omega_S(s, t)) D_{s+}^\alpha \tilde{E}(u(\cdot) - u(s), \cdot) [r] D_{q-}^{1-\alpha} \omega_{q-} [r] dr \\ &\quad - (-1)^\alpha \int_s^q D_{s+}^\alpha \omega_S(\cdot, t) [r] \tilde{E}(u(r) - u(s), \cdot) D_{q-}^{1-\alpha} \omega_{q-} [r] dr \\ &\quad + \int_s^q Df_{\tilde{E}}(\omega_S(r, t), u(r) - u(s)) (\omega_S(r, t) - \omega_S(s, t), u(r) - u(s)) \omega'(r) dr.\end{aligned}$$

Now we calculate the derivative of $f_{\tilde{E}}$:

$$\begin{aligned}Df_{\tilde{E}}(\omega_S(r, t), u(r) - u(s)) (\omega_S(r, t) - \omega_S(s, t), u(r) - u(s)) \omega'(r) \\ = (\omega_S(r, t) - \omega_S(s, t)) \tilde{E}(u(r) - u(s), \omega'(r)) + \omega_S(r, t) \tilde{E}(u(r) - u(s), \omega'(r)) \\ = \tilde{E}(u(r) - u(s), \cdot) D_2(\omega_S(t) \otimes \omega)(s, r) + \omega_S(r, t) \tilde{E} D_2(u \otimes \omega)(s, r).\end{aligned}$$

Substituting the above expression in (3.12), after applying integration by parts to the last two terms, we want to calculate $ED_{q-}^{1-\alpha}(\omega_S(t) \otimes \omega)_{q-}(s, \cdot) [r]$ and $\tilde{E} D_{q-}^{1-\alpha}(u \otimes \omega)(t)(s, \cdot)_{q-} [r]$. First, we have

$$\begin{aligned}ED_{q-}^{1-\alpha}(\omega_S(t) \otimes \omega)_{q-}(s, \cdot) [r] &= D_{q-}^{1-\alpha} E(\omega_S(t) \otimes \omega)_{q-}(s, \cdot) [r] \\ &= \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{E(\omega_S(t) \otimes \omega)(s, r) - E(\omega_S(t) \otimes \omega)(s, q)}{(q-r)^{1-\alpha}} \right. \\ &\quad \left. + (1-\alpha) \int_r^q \frac{E(\omega_S(t) \otimes \omega)(s, r) - E(\omega_S(t) \otimes \omega)(s, \theta)}{(\theta-r)^{2-\alpha}} d\theta \right) \\ &= -\frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{E(\omega_S(t) \otimes \omega)(r, q) + (\omega_S(r, t) - \omega_S(s, t)) E(\omega(q) - \omega(r))}{(q-r)^{1-\alpha}} \right. \\ &\quad \left. + (1-\alpha) \int_r^q \frac{E(\omega_S(t) \otimes \omega)(r, \theta) + (\omega_S(r, t) - \omega_S(s, t)) E(\omega(\theta) - \omega(r))}{(\theta-r)^{2-\alpha}} d\theta \right) \\ &= -ED_{q-}^{1-\alpha}(\omega_S(t) \otimes \omega) [r] + (\omega_S(r, t) - \omega_S(s, t)) ED_{q-}^{1-\alpha} \omega_{q-}(r).\end{aligned}$$

Secondly, in a similar way, by (2.17), we obtain that

$$\tilde{E} D_{q-}^{1-\alpha}(u \otimes \omega)(t)(s, \cdot)_{q-} [r] = -\tilde{E} D_{q-}^{1-\alpha}(u \otimes \omega)(t) [r] - \tilde{E}(u(r) - u(s), D_{q-}^{1-\alpha} \omega_{q-}(r)),$$

and substituting the last two expressions into (3.12) we obtain the conclusion. \square

4. GLOBAL EXISTENCE AND UNIQUENESS

We now want to find an appropriate formulation for (1.1). As we said in the Introduction, ω is a Hölder continuous function of order $\beta' \in (1/3, 1/2)$, hence the integral with integrator ω is not well defined in the classical sense. However, in what follows we will see that the two last terms in (3.2) are well defined when ω is β' -Hölder continuous. A main difficulty in that point is that for a non regular path ω we cannot expect that $(u \otimes \omega)$, which appears in the last term on (3.2), is well defined. However, we are able to overcome these problems by formulating the term $(u \otimes \omega)$ by another operator equation. This will be possible thanks to the $2\beta'$ -Hölder continuity of $(\omega \otimes_S \omega)$, as we will explain below.

We now introduce for *every* β' -Hölder continuous path ω the phase space in which we are looking for solutions to the problem (1.1):

$$W(T_1, T_2) = C_\beta([T_1, T_2]; V) \times C_{\beta+\beta'}(\Delta_{T_1, T_2}; V \otimes V)$$

for $0 \leq T_1 < T_2$, with seminorm

$$|||U||| = \|u\|_\beta + \|v\|_{\beta+\beta'}, \quad U = (u, v) \in W(T_1, T_2),$$

and such that the Chen equality holds for U , which means that for $0 \leq T_1 \leq s \leq r \leq t \leq T_2$,

$$(4.1) \quad v(s, r) + v(r, t) + (u(r) - u(s)) \otimes_V (\omega(t) - \omega(r)) = v(s, t),$$

where ω denotes a fixed β' -Hölder path with $\beta' \in (1/3, 1/2)$. Note that, when we consider for u the subset of functions with a fixed value say at T_1 , the expression $|||U|||$ generates a complete metric, see Section 2. For the metric $d_W(U_1, U_2)$ we will write $|||U_1 - U_2|||$.

When $u_1(T_1) \neq u_2(T_1)$, $W(T_1, T_2)$ becomes a Banach space if we add $|u(T_1)|$ to $|||U|||$. However, as we will see below, see Remark 4.11, it suffices to work with the seminorm $|||\cdot|||$ as we have already defined.

Recall that the spaces appearing in the definition of $W(T_1, T_2)$ as well as the corresponding norms were also introduced in Section 2.

We now give the definition of a solution to (1.1). For the sake of brevity, we assume from now on that $F = 0$. In addition, we consider solutions only on the interval $[0, 1]$.

Definition 4.1. Assume that $u_0 \in V$, and G satisfies the conditions described on Section 2. A mild solution of (1.1) is a pair $U = (u, v) \in W(0, 1)$ satisfying

$$(4.2) \quad \begin{aligned} u(t) &= S(t)u_0 + (-1)^\alpha \int_0^t \hat{D}_{0+}^\alpha(S(t-\cdot)G(u(\cdot)))[r]D_{t-}^{1-\alpha}\omega_{t-}[r]dr \\ &\quad - (-1)^{2\alpha-1} \int_0^t D_{0+}^{2\alpha-1}(S(t-\cdot)DG(u(\cdot)))[r]D_{t-}^{1-\alpha}\mathcal{D}_{t-}^{1-\alpha}v[r]dr, \end{aligned}$$

$$(4.3) \quad \begin{aligned} v(s, t) &= \int_s^t (S(\xi - s) - \text{id})u(s) \otimes_V d\omega(\xi) \\ &\quad - (-1)^\alpha \int_s^t \hat{D}_{s+}^\alpha G(u(\cdot))[r]D_{t-}^{1-\alpha}(\omega \otimes_S \omega)(\cdot, t)_{t-}[r]dr \\ &\quad + (-1)^{2\alpha-1} \int_s^t D_{s+}^{2\alpha-1}DG(u(\cdot))[r]D_{t-}^{1-\alpha}\mathcal{D}_{t-}^{1-\alpha}(u \otimes (\omega \otimes_S \omega)(t))[r]dr, \end{aligned}$$

for $0 \leq s < t \leq 1$. The term $(u \otimes (\omega \otimes_S \omega)(t))(s, t)$ can be defined by the right hand side of (3.11) where we replace $u \otimes \omega$ by v , that is, for $\tilde{E} \in L_2(V \otimes V, V_\delta)$,

$$(4.4) \quad \begin{aligned} \tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, q) &= - \int_s^q \hat{D}_{s+}^\alpha \omega_S(\cdot, t) \tilde{E}(u(\cdot) - u(s), \cdot)[r]D_{q-}^{1-\alpha}\omega_{q-}[r]dr \\ &\quad + (-1)^{\alpha-1} \int_s^q D_{s+}^{2\alpha-1} \tilde{E}(u(\cdot) - u(s), \cdot)[r]D_{q-}^{1-\alpha}\mathcal{D}_{q-}^{1-\alpha}(\omega_S(t) \otimes \omega)[r]dr \\ &\quad + (-1)^{\alpha-1} \int_s^q D_{s+}^{2\alpha-1} \omega_S(\cdot, t)[r] \tilde{E}D_{q-}^{1-\alpha}\mathcal{D}_{q-}^{1-\alpha}v[r]dr, \end{aligned}$$

where $(\omega_S(t) \otimes \omega)$ is defined for $s \leq \tau \leq t$, $E \in L_2(V, V_\delta)$ as

$$(4.5) \quad E(\omega_S(t) \otimes \omega)(s, \tau) = \omega_S(\tau, t) \int_s^\tau S(\tau - r) E d\omega(r) + (-1)^{-\alpha} E(\omega \otimes_S \omega)(s, \tau) - \omega_S(s, t) E(\omega(\tau) - \omega(s)).$$

Remark 4.2. The notation of the tensor given by (4.4) has been inherited by its analogous counterpart in the finite-dimensional case, see [17]. However, we would like to point that, despite its name, the tensor $u \otimes (\omega \otimes_S \omega)$ depends on v as can be seen in its definition.

We denote the right hand side of the system (4.2)-(4.3) by $\mathcal{T}(U) = (\mathcal{T}_1(U), \mathcal{T}_2(U))$. For the *sum of both* integrals in (4.2) over an interval $[s, t]$ we use the abbreviation

$$\int_s^t S(t-r)G(u(r))d\omega(r).$$

Observe that in the previous definition, the equations for the second component v as well as $(u \otimes (\omega \otimes_S \omega))$ can be regarded as the generalizations to the non-regular case of the equations we have in the regular case for $(u \otimes \omega)$ and $(u \otimes (\omega \otimes_S \omega))$ (see Section 3). Due to Lemma 6.5 it is therefore natural to assume that v given by (4.3) satisfies (4.1).

Remark 4.3. *We have defined the solution with respect to the time interval $[0, 1]$. Similar we can define a solution on $[T_1, T_2]$, $0 < T_1 < T_2$, when the initial condition is given at T_1 and the space for the solution is $W(T_1, T_2)$.*

In order to establish the existence and uniqueness of solutions to system (4.2)-(4.3), we set the following Hypothesis **H**:

- (1) Assume that S is an analytic semigroup with generator A on the separable Hilbert space V . We assume that $-A$ is positive and symmetric and that A has a compact inverse, generating a complete base of eigenelements $(e_i)_{i \in \mathbb{N}}$ of V . We also need to assume for the associated spectrum $(\lambda_i)_{i \in \mathbb{N}}$ of A that

$$(4.6) \quad \sum_i \lambda_i^{-2\delta} < \infty,$$

where $\delta \in [0, 1]$ is an appropriate parameter which rank value will be determined later.

Assume $G : V \mapsto L_2(V, V_\delta)$ is a three times Fréchet differentiable mapping with bounded derivatives as in Lemma 2.3, such that $DG(\cdot) \in L_2(V \otimes V, V_\delta)$.

- (2) Suppose that $1/3 < H \leq 1/2$ and $1/3 < \beta < H$. Suppose that there is an α such that $1 - \beta < \alpha < 2\beta$, $\alpha < \frac{\beta+1}{2}$ and $-3\beta + \alpha + H > 0$. Hence we can assume that $-2\beta + 2\alpha + H > 1$.
- (3) Let $\omega \in C_{\beta'}([0, 1]; V)$ for any $\beta < \beta' < H$ such that the last two inequalities in item (2) hold if we replace H by β' .
- (4) Suppose that $(\omega \otimes_S \omega) \in C_{2\beta'}(\Delta_{0,1}; L_2(L_2(V, V_\delta), V \otimes V))$, and in particular the mapping $L_2(V, V_\delta) \ni E \mapsto E(\omega \otimes_S \omega)(s, t)$ is linear and bounded for fixed $(s, t) \in \Delta_{0,1}$. In addition the Chen equality holds in the following way: for $E \in L_2(V, V_\delta)$, $0 \leq s \leq r \leq t \leq 1$ we have

$$E(\omega \otimes_S \omega)(s, r) + E(\omega \otimes_S \omega)(r, t) + (-1)^{-\alpha} \omega_S(r, t) \int_s^r S(r-q)E d\omega(q) = E(\omega \otimes_S \omega)(s, t).$$

- (5) Let $(\omega^n)_{n \in \mathbb{N}}$ be a sequence of piecewise smooth functions with values in V such that $((\omega^n \otimes_S \omega^n))_{n \in \mathbb{N}}$ is defined by (3.4). Assume then that for any $\beta' < H$ the sequence $((\omega^n, (\omega^n \otimes_S \omega^n(\cdot, \cdot))))_{n \in \mathbb{N}}$ converges to $(\omega, (\omega \otimes_S \omega(\cdot, \cdot)))$ in $C_{\beta'}([0, 1]; V) \times C_{2\beta'}(\Delta_{0,1}; L_2(L_2(V, V_\delta), V \otimes V))$.

Remark 4.4. (i) In item (1) above note that the negativeness of the operator A is not a restriction, since otherwise we could consider $A - \text{cid}$ and $F + \text{cid}$ instead of the respective A and F , with c a positive constant.

(ii) The integrals $\int_s^r S(r-q)E d\omega(q)$ and $\omega_S(r, t) = (-1)^\alpha \int_r^t S(\xi-r) \cdot \otimes_V d\omega(\xi)$ in item (4) are well defined in the Weyl sense if S and ω satisfy the above properties, see Lemma 4.5 below.

(iii) We have that

$$(4.7) \quad |D_{t-}^{1-\alpha} \omega_{t-}[r]| \leq c \|\omega\|_{\beta'} (t-r)^{\alpha+\beta'-1},$$

which follows easily from the Hölder condition on ω , item (3) above.

(iv) To interpret (1.1) as a stochastic partial differential equation we can assume that ω is given by a fractional Brownian motion with Hurst parameter $H \in (1/3, 1/2]$.

From now on, we will use c or C to denote a generic positive constant which value is not so important and that may change from line to line. That constant may depend on parameters, for instance, it may depend on ω .

At this point, it is crucial to stress that it is possible to give sense to all integrals appearing in our definition of solution, namely, to every integral in (4.2)-(4.3). As it was shown in Section 2 and Section 3, in order to do

that it is necessary to make use of the L_2 spaces of Hilbert–Schmidt operators. Hence, for instance, for the last integral on the right hand side of (4.3), we need $DG(u)$ to belong to $L_2(V \otimes V, V_\delta)$ and $(u \otimes (\omega \otimes_S \omega))$ to be an element in $L_2(L_2(V \otimes V, V_\delta), V \otimes V)$.

Note that in (4.2), the operators $\hat{D}_{0+}^{\alpha-1}(S(t - \cdot)G(u(\cdot)))[r]$ and $D_{0+}^{2\alpha-1}(S(t - \cdot)DG(u(\cdot)))[r]$ are applied, respectively, to an element in V and $V \otimes V$. However, in (4.3) this is different. There the operators $(\omega \otimes_S \omega)$ and $(u \otimes (\omega \otimes_S \omega))$, considered to be contained in $L_2(L_2(V, V_\delta), V \otimes V)$ and $L_2(L_2(V \otimes V, V_\delta), V \otimes V)$, are applied to the elements $\hat{D}_{s+}^{\alpha}G(u(\cdot))[r]$ and $D_{s+}^{2\alpha-1}DG(u(\cdot))[r]$. Note that this last regularity property of the tensor $(u \otimes (\omega \otimes_S \omega))$ is not part of the Hypothesis **H** and will be analyzed in Lemma 4.6.

Now we establish some properties of ω_S , which are consequences of (2.10) due to the regularity of the semigroup.

Lemma 4.5. *Under the Hypothesis **H** the following statements hold:*

(i) *For $0 \leq s \leq r \leq t \leq 1$, $e \in V$ and $\beta' < \beta'' < H$ we have that*

$$\|\omega_S(r, t)e - \omega_S(s, t)e\| \leq c|r - s|^{\beta'}(\|\omega\|_{\beta'} + \|\omega\|_{\beta''})|e|, \quad \|\omega_S(s, t)e\| \leq c|t - s|^{\beta'}\|\omega\|_{\beta'}|e|.$$

(ii) *The mapping*

$$E \in L_2(V, V_\delta) \mapsto I(E) := \int_s^t S(t - r)Ed\omega(r)$$

is in $L_2(L_2(V, V_\delta), V)$, with norm bounded by $c\|\omega\|_{\beta'}(t - s)^{\beta'}$.

Proof. Let $\beta' < \beta'' < H$ such that for $\alpha' < \alpha'' < 1$ we have

$$\beta' + \alpha'' < 1 < \beta'' + \alpha'.$$

Then

$$(4.8) \quad \begin{aligned} & \omega_S(r, t)e - \omega_S(s, t)e \\ &= (-1)^\alpha \int_r^t (S(\xi - r)e - S(\xi - s)e) \otimes_V d\omega(\xi) - (-1)^\alpha \int_s^r S(\xi - s)e \otimes_V d\omega(\xi). \end{aligned}$$

Taking Lemma 2.1 into account we obtain

$$\begin{aligned} |D_{r+}^{\alpha'}(S(\cdot - r)e - S(\cdot - s)e)[\xi]| &\leq c \left(\frac{(r - s)^{\beta'}}{(\xi - r)^{\alpha' + \beta'}} + \alpha' \int_r^\xi \frac{(r - s)^{\beta'}(\xi - q)^{\alpha''}}{(\xi - q)^{1 + \alpha'}(q - r)^{\alpha'' + \beta'}} dq \right) |e| \\ &\leq c(r - s)^{\beta'}(\xi - r)^{-\alpha' - \beta'}|e|. \end{aligned}$$

Moreover, we can write (4.7) in the following way

$$|D_{t-}^{1 - \alpha'}\omega_{t-}[\xi]| \leq c\|\omega\|_{\beta''}(t - \xi)^{\alpha' + \beta'' - 1}.$$

Hence, by applying Lemma 6.1, the first integral on the right hand side of (4.8) is bounded by $c|r - s|^{\beta'}\|\omega\|_{\beta''}|e|$.

Furthermore, for the last term in (4.8) and $1 - \beta' < \alpha' < \alpha'' < 1$, in a similar way we obtain

$$\begin{aligned} \int_s^r \|D_{s+}^{\alpha'}S(\cdot - s)e[\xi] \otimes_V D_{r-}^{1 - \alpha'}\omega_{r-}[\xi]\| d\xi &\leq \int_s^r |D_{s+}^{\alpha'}S(\cdot - s)e[\xi]| |D_{r-}^{1 - \alpha'}\omega_{r-}[\xi]| d\xi \\ &\leq c\|\omega\|_{\beta'}|e| \int_s^r \left((\xi - s)^{-\alpha'} + \int_s^\xi \frac{(\xi - q)^{\alpha''}}{(q - s)^{\alpha''}(\xi - q)^{\alpha' + 1}} dq \right) (r - \xi)^{\alpha' + \beta' - 1} d\xi \\ &\leq c\|\omega\|_{\beta'}|r - s|^{\beta'}|e|. \end{aligned}$$

The second conclusion of (i) follows directly from the last inequality taking $r = t$, therefore the proof of (i) is complete.

Now we prove (ii). To see that $I(E)$ is well-defined, it is enough to check that $r \mapsto S(t - r)E$ is locally Hölder continuous (see (2.10) and Remark 2.5). In fact, for $e \in V_\delta$, for β' from Hypothesis **H** we can choose $0 < \alpha < \alpha'$ with $\alpha + \beta' - \alpha' > 0$ such that

$$|(S(t - r) - S(t - q))e| \leq c(r - q)^{\alpha'}|(-A)^{\alpha'}S(t - r)e| \leq c(r - q)^{\alpha'}(t - q)^{-\alpha' + \delta}|e|_{V_\delta},$$

and therefore, according to (2.10), Remark 2.5 and (4.7), for $e \in V_\delta$, in particular

$$|I(e)| \leq (t - s)^{\beta'}|e|_{V_\delta}\|\omega\|_{\beta'}.$$

Now we study

$$\begin{aligned} \|I(\cdot)\|_{L_2(L_2(V, V_\delta), V)} &= \left\| \int_s^t S(t-r) \cdot d\omega(r) \right\|_{L_2(L_2(V, V_\delta), V)} \\ &\leq c \int_s^t \|D_{s+}^\alpha S(t-\cdot)[r]\|_{L(V)} \cdot \|D_{t-}^{1-\alpha} \omega_{t-}[r]\|_{L_2(L_2(V, V_\delta), V)} dr \leq c \|\omega\|_{\beta'} (t-s)^{\beta'}. \end{aligned}$$

It is easily seen that the integrand $D_{s+}^\alpha S(t-\cdot)[r] \cdot D_{t-}^{1-\alpha} \omega_{t-}[r]$ of this integral with values in $L_2(L_2(V, V_\delta), V)$ is weakly measurable with respect to the separable Hilbert space $L_2(L_2(V, V_\delta), V)$ such that by Pettis' theorem the integrand is measurable. The norm of

$$E \mapsto ED_{t-}^{1-\alpha} \omega_{t-}[r]$$

in $L_2(L_2(V, V_\delta), V)$ is just $(\sum_i \lambda_i^{-2\delta})^{\frac{1}{2}} \|D_{t-}^{1-\alpha} \omega_{t-}[r]\|$ which can be estimated by (4.6) and (4.7). \square

Lemma 4.6. *Under the Hypothesis **H**, for $0 \leq s \leq r \leq q \leq t \leq 1$, $\tilde{E} \in L_2(V \otimes V, V_\delta)$ and $U = (u, v) \in W(0, 1)$ for the mapping*

$$L_2(V \otimes V, V_\delta) \ni \tilde{E} \mapsto \tilde{E}(u \otimes (\omega \otimes_S \omega)(t))(s, q) \in L_2(L_2(V \otimes V, V_\delta), V \otimes V)$$

it holds

$$(4.9) \quad \|(u \otimes (\omega \otimes_S \omega)(t))(s, q)\|_{L_2(L_2(V \otimes V, V_\delta), V \otimes V)} \leq c \|U\| (q-s)^{\beta+\beta'} (t-s)^{\beta'}.$$

Moreover, the expression $(u \otimes (\omega \otimes_S \omega))$ satisfies the Chen equality (3.10). In addition,

$$(4.10) \quad \|D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} (u \otimes (\omega \otimes_S \omega)(t))[r]\|_{L_2(L_2(V \otimes V, V_\delta), V \otimes V)} \leq c \|U\| (t-r)^{\beta+2\beta'+2\alpha-2}.$$

Proof. Let us consider separately the three terms of $(u \otimes (\omega \otimes_S \omega)(t))(s, q)$ given in (4.4). For every integral of $(u \otimes (\omega \otimes_S \omega)(t))(s, q)$ we shall firstly deal with the existence of the integral; secondly we will prove that these integrals define elements in $L_2(L_2(V \otimes V, V_\delta), V \otimes V)$, and finally, we will obtain adequate estimates of their norms in the mentioned space.

We start with

$$I_1(\tilde{E}) := \int_s^q D_{s+}^{2\alpha-1} \omega_S(\cdot, t)[r] D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} \tilde{E} v[r] dr.$$

We show the existence of I_1 . In fact this integral is well defined since the factors in the integrand are Hölder-continuous. Let us look at this statement with more details: firstly we can exchange \tilde{E} and the fractional derivatives $D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha}$ (see (4.12) below), so that the two factors become $D_{s+}^{2\alpha-1} \omega_S(\cdot, t) \tilde{E}[r]$ and $D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} v[r]$. By assumption $v \in C_{\beta+\beta'}(\Delta_{0,1}; V \otimes V)$ and thus (2.20) holds. Furthermore, Lemma 4.5 guarantees that $\omega_S(\cdot, t) \tilde{E} \in C_{\beta'}([s, t]; L_2(V \otimes V, V \otimes V))$. This means that the factors satisfy the conditions for the existence of the integral (see section 2 to find the conditions under which (2.19) is well defined).

For a fixed $v \in V \otimes V$ the mapping

$$(4.11) \quad L_2(V \otimes V, V_\delta) \ni \tilde{E} \mapsto \tilde{E} v$$

is in $L_2(L_2(V \otimes V, V_\delta), V)$. The norm with respect to this space is $(\sum_{i=1}^\infty \lambda_i^{-2\delta})^{\frac{1}{2}} \|v\|$. Then, since from Lemma 4.5 (i) $D_{s+}^{2\alpha-1} \omega_S(r, t)$ is in $L(V, V \otimes V)$, $\tilde{E} \mapsto I_1(\tilde{E})$ is a mapping in $L_2(L_2(V \otimes V, V_\delta), V \otimes V)$.

In particular we consider the above integral with an integrand having values in the separable Hilbert space $L_2(L_2(V \otimes V, V_\delta), V \otimes V)$. It is not hard to see by Pettis' Theorem that the integrand of this integral is weakly measurable and hence measurable as a mapping from $[s, q]$ into $L_2(L_2(V \otimes V, V_\delta), V \otimes V)$. Moreover, we have

$$\begin{aligned} \|I_1\|_{L_2(L_2(V \otimes V, V_\delta), V \otimes V)} &\leq \int_s^q \left\| D_{s+}^{2\alpha-1} \omega_S(\cdot, t)[r] D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} \cdot v[r] \right\|_{L_2(L_2(V \otimes V, V_\delta), V \otimes V)} dr \\ &\leq \int_s^q \|D_{s+}^{2\alpha-1} \omega_S(\cdot, t)[r]\|_{L(V, V \otimes V)} \|D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} \cdot v[r]\|_{L_2(L_2(V \otimes V, V_\delta), V \otimes V)} dr. \end{aligned}$$

In order to estimate the second factor in the integrand of I_1 , we note that

$$(4.12) \quad D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} \tilde{E}v[r] = \tilde{E} D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} v[r]$$

which follows easily simply exchanging the integrals in the definition of the fractional derivatives and \tilde{E} . Under the Hypothesis **H**, for $r \in (0, 1)$ and $v \in C_{\beta+\beta'}(\Delta_{0,1}; V \otimes V)$,

$$\|D_{t-}^{1-\alpha} v[r]\| \leq c \|v\|_{\beta+\beta'} (t-r)^{\beta+\beta'+\alpha-1}.$$

Then, thanks to (4.1), for $r \in (0, 1)$ and $U \in W(0, 1)$, we obtain

$$(4.13) \quad \|D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} v[r]\| \leq c (\|v\|_{\beta+\beta'} + \|u\|_{\beta} \|\omega\|_{\beta'}) (t-r)^{\beta+\beta'+2\alpha-2}.$$

In fact, (4.11) together with (4.13) immediately implies that

$$\|D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} \cdot v[r]\|_{L_2(L_2(V \otimes V, V_{\delta}), V)} \leq c (\|v\|_{\beta+\beta'} + \|u\|_{\beta} \|\omega\|_{\beta'}) (q-r)^{2\alpha+\beta+\beta'-2}$$

and by Lemma 4.5 (i), for $H > \beta'' > \beta'$,

$$\|D_{s+}^{2\alpha-1} \omega_S(\cdot, t)[r]\|_{L(V, V \otimes V)} \leq c \left(\frac{(t-r)^{\beta'}}{(r-s)^{2\alpha-1}} + \int_s^r \frac{(r-q)^{\beta'}}{(r-q)^{2\alpha}} dq \right) (\|\omega\|_{\beta'} + \|\omega\|_{\beta''}).$$

Combining the previous estimates we can conclude

$$(4.14) \quad \left\| \int_s^q D_{s+}^{2\alpha-1} \omega_S(\cdot, t)[r] D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} \cdot v[r] dr \right\|_{L_2(L_2(V \otimes V, V_{\delta}), V \otimes V)} \\ \leq c (\|v\|_{\beta+\beta'} + \|u\|_{\beta} \|\omega\|_{\beta'}) (\|\omega\|_{\beta'} + \|\omega\|_{\beta''}) (t-s)^{\beta'} (q-s)^{\beta+\beta'}.$$

Next we deal with

$$I_2(\tilde{E}) := \int_s^q \hat{D}_{s+}^{\alpha} \omega_S(\cdot, t) \tilde{E}(u(\cdot) - u(s), \cdot)[r] D_{q-}^{1-\alpha} \omega_{q-}[r] dr.$$

According to Taylor expansion we calculate the norm of the compensated fractional derivative $\|\hat{D}_{s+}^{\alpha} \omega_S(\cdot, t) \tilde{E}(u(\cdot) - u(s), \cdot)[r]\|_{L_2(V, V \otimes V)}$ in the following manner:

$$(4.15) \quad \|\hat{D}_{s+}^{\alpha} \omega_S(\cdot, t) \tilde{E}(u(\cdot) - u(s), \cdot)[r]\|_{L_2(V, V \otimes V)} \leq \left\| \frac{\omega_S(r, t) \tilde{E}(u(r) - u(s), \cdot)}{(r-s)^{\alpha}} \right\|_{L_2(V, V \otimes V)} \\ + \alpha \left\| \int_s^r \frac{(\omega_S(r, t) - \omega_S(\theta, t)) \tilde{E}(u(r) - u(\theta), \cdot)}{(r-\theta)^{1+\alpha}} d\theta \right\|_{L_2(V, V \otimes V)} \\ \leq c ((t-r)^{\beta'} (r-s)^{\beta-\alpha} + (r-s)^{\beta'+\beta-\alpha}) \|u\|_{\beta} \|\tilde{E}\|_{L_2(V \otimes V, V_{\delta})}.$$

The last inequality follows thanks to the regularity of ω_S , see Lemma 4.5 (i). We obtain that $I_2(\tilde{E})$ is well defined for every $\tilde{E} \in L_2(V \otimes V, V_{\delta})$. In a similar way as we have already done with I_1 , we can interpret I_2 as an element in $L_2(L_2(V \otimes V, V_{\delta}), V \otimes V)$, since

$$\tilde{E} \in L_2(V \otimes V, V_{\delta}) \mapsto \tilde{E}(u(r) - u(s), \cdot) D_{q-}^{1-\alpha} \omega_{q-}[r]$$

is in $L_2(L_2(V \otimes V, V_{\delta}), V)$, where the norm of this mapping is given by $(\sum_{i=1}^{\infty} \lambda_i^{-2\delta})^{\frac{1}{2}} \|u(r) - u(s)\| D_{q-}^{1-\alpha} \omega_{q-}[r]$. By the regularity conclusions for $\omega_S(\cdot, t)$ we obtain that the mapping $\tilde{E} \in L_2(V \otimes V, V_{\delta}) \mapsto I_2(\tilde{E})$ is in $L_2(L_2(V \otimes V, V_{\delta}), V \otimes V)$.

Again, by Pettis' Theorem the integrand of I_2 is measurable. Lemma 4.5 (i) and (4.7) gives

$$\|I_2\|_{L_2(L_2(V \otimes V, V_{\delta}), V \otimes V)} \leq \int_s^q \left\| \hat{D}_{s+}^{\alpha} \omega_S(\cdot, t) \cdot (u(\cdot) - u(s))[r] D_{q-}^{1-\alpha} \omega_{q-}[r] \right\|_{L_2(L_2(V \otimes V, V_{\delta}), V \otimes V)} dr \\ \leq c \|u\|_{\beta} \|\omega\|_{\beta'} (\|\omega\|_{\beta''} + \|\omega\|_{\beta'}) (q-s)^{\beta+\beta'} (t-s)^{\beta'}.$$

for $H > \beta'' > \beta'$.

Now we estimate the third term:

$$I_3(\tilde{E}) := \int_s^q D_{s+}^{2\alpha-1} \tilde{E}(u(\cdot) - u(s), \cdot)[r] D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} (\omega_S(t) \otimes \omega)[r] dr := I_{31} + I_{32} + I_{33},$$

where these integrals correspond to the three different terms coming from the expression of $(\omega_S(t) \otimes \omega)$ given in (4.5).

We also need that for a fixed $u = \sum_j \hat{u}_j e_j \in V$

$$\sum_i |\tilde{E}(u, e_i)|_{V_\delta}^2 = \sum_i \left| \sum_j \tilde{E}(\hat{u}_j e_j, e_i) \right|_{V_\delta}^2 \leq \sum_i \sum_j \hat{u}_j^2 \sum_j |\tilde{E}(e_j, e_i)|_{V_\delta}^2 = |u|^2 \|\tilde{E}\|_{L_2(V \otimes V, V_\delta)}^2$$

such that

$$(4.16) \quad L_2(V \otimes V, V_\delta) \ni \tilde{E} \mapsto \tilde{E}(u, \cdot) \in L_2(V, V_\delta)$$

is contained in $L(L_2(V \otimes V, V_\delta), L_2(V, V_\delta))$ with norm $|u|$.

Let us start with the first integral I_{31} .

$$I_{31}(\tilde{E}) = \int_s^q D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} \left(\omega_S(\cdot, t) \int_\cdot S(\cdot - \xi) d\omega(\xi) \right) [r] D_{s+}^{2\alpha-1} \tilde{E}(u(\cdot) - u(s), \cdot) [r] dr$$

where, for $E \in L_2(V, V_\delta)$, $\omega_S(\cdot, t) \int_\cdot S(\cdot - \xi) E d\omega(\xi)$ is the mapping $\Delta_{0,1} \ni (s, q) \mapsto \omega_S(q, t) \int_s^q S(q - \xi) E d\omega(\xi) \in V \otimes V$. This integral exists because $E(\cdot, D_{q-}^{1-\alpha} \omega_{q-}) \in L_2(V, V_\delta)$, and by Lemma 4.5 (i) and (ii) for $H > \beta'' > \beta'$, we have that

$$\|\omega_S(\tau, t) \int_s^\tau S(\tau - r) \tilde{E}(\cdot, d\omega)\|_{L_2(V, V \otimes V)} \leq c(t - \tau)^{\beta'} (\tau - s)^{\beta'} \|\tilde{E}\|_{L_2(V \otimes V, V_\delta)} \|\omega\|_{\beta'} (\|\omega\|_{\beta''} + \|\omega\|_{\beta'}),$$

and therefore,

$$\left\| D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} \left(\omega_S(\cdot, t) \int_\cdot S(\cdot - \xi) d\omega(\xi) \right) [r] \right\| \leq c(t - q)^{\beta'} (q - r)^{\beta' + 2\alpha - 2},$$

such that I_{31} can be interpreted as the integral (2.19).

Now notice that, similar to (4.12), it can be checked that it is possible to exchange $D_{s+}^{2\alpha-1}$ and \tilde{E} and therefore, by (4.16), the mapping $\mathcal{E}_1 : \tilde{E} \mapsto \tilde{E}(D_{s+}^{2\alpha-1}(u(r) - u(s)), \cdot)$ is in $L(L_2(V \otimes V, V_\delta), L_2(V, V_\delta))$ with

$$(4.17) \quad \|\mathcal{E}_1\|_{L(L_2(V \otimes V, V_\delta), L_2(V, V_\delta))} = |D_{s+}^{2\alpha-1}(u(r) - u(s))|.$$

Moreover, thanks to Lemma 3.4 (ii), we know that $L_2(V, V_\delta) \ni E \mapsto \int_s^\tau S(\tau - r) E d\omega(r)$ is in $L_2(L_2(V, V_\delta), V)$. In particular, we also have

$$(4.18) \quad \mathcal{E}_2 : L_2(V, V_\delta) \ni E \mapsto D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} \left(\omega_S(\cdot, t) \int_\cdot S(\cdot - \xi) E d\omega(\xi) \right) [r]$$

is in $L_2(L_2(V, V_\delta), V \otimes V)$. Therefore, the composition $\mathcal{E}_2 \circ \mathcal{E}_1$ of the mappings in (4.17) and (4.18) allows us to conclude that the integral is Hilbert-Schmidt and the mapping $\tilde{E} \mapsto I_{31}(\tilde{E})$ is in $L_2(L_2(V \otimes V, V_\delta), V \otimes V)$.

We need Pettis' Theorem to ensure that the integrand having values in $L_2(L_2(V \otimes V, V_\delta), V \otimes V)$ is measurable. Hence

$$\begin{aligned} & \|I_{31}\|_{L_2(L_2(V \otimes V, V_\delta), V \otimes V)} \\ & \leq \int_s^q \left\| D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} \left(\omega_S(\cdot, t) \int_\cdot S(\cdot - \xi) d\omega(\xi) \right) [r] D_{s+}^{2\alpha-1} \cdot (u(\cdot) - u(s), \cdot) [r] \right\|_{L_2(L_2(V \otimes V, V_\delta), V \otimes V)} dr \\ & \leq \int_s^q \|D_{q-}^{1-\alpha} \mathcal{D}_{q-}^{1-\alpha} \left(\omega_S(\cdot, t) \int_\cdot S(\cdot - \xi) \cdot d\omega(\xi) \right) [r]\|_{L_2(L_2(V, V_\delta), V \otimes V)} |D_{s+}^{2\alpha-1}(u(\cdot) - u(s)) [r]| dr \\ & \leq c \|\omega\|_{\beta'} (\|\omega\|_{\beta''} + \|\omega\|_{\beta'}) \|u\|_{\beta} (t - s)^{\beta'} (q - s)^{\beta + \beta'}, \end{aligned}$$

where $H > \beta'' > \beta'$. Finally, notice that by Hypothesis **H**, item (4), the integral I_{32} can be estimated in a similar manner. For I_{33} we can find an estimate by the method used to estimate I_1 or I_2 .

Collecting the estimates for I_1 , I_2 and I_3 we get (4.9).

The Chen property follows since, as we proved in Section 3, $(u \otimes (\omega \otimes_S \omega))$ satisfies (3.10) in the regular case. It suffices therefore to consider an approximation argument to show this property in the non-smooth case.

Moreover, the previous estimate for $(u \otimes (\omega \otimes_S \omega))$ implies (4.10) similarly to (2.20) (see Lemma 6.3 in [17] for more details). To get it we want to emphasize that it is necessary to use the Chen equality (3.10). \square

The following estimates are crucial for the existence of a solution.

Lemma 4.7. *Suppose Hypothesis **H** holds. Then there exists $C > 0$ such that for $T \in [0, 1]$ and $U \in W(0, T)$ we have*

$$(i) \text{ If } u_0 \in V_\delta \text{ with } \delta \geq \beta, \\ (4.19) \quad \|\mathcal{T}_1(U)\|_\beta \leq CT^{\delta-\beta}|u_0|_{V_\delta} + CT^{\beta'-\beta}(1 + T^{2\beta}|||U|||^2).$$

(ii) If $u_0 \in V_\delta$,

$$|\mathcal{T}_1(U)(T)|_{V_\delta} \leq |u_0|_{V_\delta} + CT^{\beta'}(1 + T^{2\beta}|||U|||^2).$$

Remark 4.8. *We want to stress that the second term in both of the above inequalities stem from the integral term of (4.2), while the first term is an estimate of $\|S(\cdot)u_0\|_\beta$ and $|S(T)u_0|_{V_\delta}$, respectively.*

The proof of this result is rather technical for which we have preferred to present it in the Appendix.

In the following result, we obtain the corresponding estimate for $\mathcal{T}_2(U)(s, t)$, which has been defined as the right hand side of equation (4.3).

Lemma 4.9. *Suppose Hypothesis **H** holds. If in addition $u_0 \in V_\delta$ for $\delta \geq \beta$, then there exists $C > 0$ such that for $T \in [0, 1]$ and $U \in W(0, T)$ we have*

$$\|\mathcal{T}_2(U)\|_{\beta+\beta'} \leq C(T^{\beta'-\beta}(1 + T^{2\beta}|||U|||^2) + T^{\delta-\beta}|u_0|_{V_\delta}).$$

Proof. Let us denote $\mathcal{T}_2(U)(s, t) =: B_1(s, t) + B_2(s, t) + B_3(s, t)$, corresponding to the three different addends of (4.3).

For B_1 we can consider the following splitting:

$$\begin{aligned} B_1(s, t) &= \int_s^t (S(\xi - s) - \text{id})u(s) \otimes_V d\omega(\xi) \\ &= \int_s^t (S(\xi) - S(s))u_0 \otimes_V d\omega(\xi) \\ &\quad + \int_s^t (S(\xi - s) - \text{id}) \left(\int_0^s S(s - r)G(u(r))d\omega(r) \right) \otimes_V d\omega(\xi) \\ &=: B_{11}(s, t) + B_{12}(s, t). \end{aligned}$$

B_1 can be interpreted in Weyl's sense thanks to the regularity of its integrand, which means that

$$B_{11}(s, t) = (-1)^\alpha \int_s^t D_{s+}^\alpha ((S(\cdot) - S(s))u_0)[\xi] \otimes_V D_{t-}^{1-\alpha} \omega_{t-}[\xi] d\xi.$$

For $\alpha < \mu < 1 + \delta$ and $s > 0$, applying (2.1) and (2.2) we have

$$\begin{aligned} |D_{s+}^\alpha ((S(\cdot) - S(s))u_0)[\xi]| &\leq \frac{1}{\Gamma(1-\alpha)} \left(\frac{|(S(\xi) - S(s))u_0|}{(\xi - s)^\alpha} + \alpha \int_s^\xi \frac{|(S(\xi) - S(q))u_0|}{(\xi - q)^{1+\alpha}} dq \right) \\ &\leq c \left((\xi - s)^{\delta-\alpha} + \int_s^\xi \frac{(\xi - q)^\mu}{(\xi - q)^{1+\alpha}(q - s)^{\mu-\delta}} dq \right) |u_0|_{V_\delta} \\ &\leq c(\xi - s)^{\delta-\alpha} |u_0|_{V_\delta}. \end{aligned}$$

From the last inequality and (4.7), for $s > 0$, it follows that

$$|B_{11}(s, t)| \leq c\|\omega\|_{\beta'}|u_0|_{V_\delta} \int_s^t (\xi - s)^{\delta-\alpha}(t - \xi)^{\beta'+\alpha-1} d\xi,$$

which immediately implies

$$\|B_{11}\|_{\beta+\beta'} \leq C|u_0|_{V_\delta} T^{\delta-\beta}.$$

Besides, note that

$$\begin{aligned} & D_{s+}^\alpha \left((S(\cdot - s) - \text{id}) \int_0^s S(s-r)G(u(r))d\omega(r) \right) [\xi] \\ & \leq C \left(\frac{(S(\xi - s) - \text{id})(\int_0^s S(s-r)G(u(r))d\omega(r))}{(\xi - s)^\alpha} + \int_s^\xi \frac{\int_0^s (S(\xi - r) - S(q - r))G(u(r))d\omega(r)}{(\xi - q)^{1+\alpha}} dq \right). \end{aligned}$$

For the second expression on the right hand side, by Lemma 4.7 (ii) for $\alpha < \mu < 1 + \delta$, with $\mu \geq \delta$, it holds

$$\begin{aligned} & \int_s^\xi \frac{\int_0^s |(S(\xi - q) - \text{id})S(q-r)G(u(r))d\omega(r)|}{(\xi - q)^{1+\alpha}} dq \\ & \leq \int_s^\xi \frac{|A^{\mu-\delta}S(q-s)|}{(\xi - q)^{1+\alpha-\mu}} \left| \int_0^s S(s-r)G(u(r))d\omega(r) \right|_{V_\delta} dq \\ & \leq C(1 + s^{2\beta}|||U|||^2)s^{\beta'} \int_s^\xi \frac{(q-s)^{\delta-\mu}}{(\xi - q)^{1+\alpha-\mu}} dq \leq C(1 + s^{2\beta}|||U|||^2)s^{\beta'}(\xi - s)^{\delta-\alpha}, \end{aligned}$$

then in particular we have by Remark 4.8

$$\|B_{12}\|_{\beta+\beta'} \leq CT^{\beta'+\delta-\beta}(1 + T^{2\beta}|||U|||^2) \leq CT^{\beta'-\beta}(1 + T^{2\beta}|||U|||^2).$$

Finally, the same estimate follows for B_2 and B_3 . In order to see this, note that B_2 and B_3 can be considered to be, respectively, the first and second integral on the right hand side of (2.18), and then, evaluating respectively the Hilbert-Schmidt operators $D_{t-}^{1-\alpha}(\omega \otimes_S \omega)(\cdot, t)_{t-}[r]$ and $D_{t-}^{1-\alpha}\mathcal{D}_{t-}^{1-\alpha}(u \otimes (\omega \otimes_S \omega)(t))[r]$ we can obtain

$$\begin{aligned} \|B_2\|_{\beta+\beta'} & \leq CT^{\beta'-\beta}(1 + T^{2\beta}|||U|||^2), \\ \|B_3\|_{\beta+\beta'} & \leq C'T^{\beta'-\beta}(T^\beta|||U||| + T^{2\beta}|||U|||^2) \leq CT^{\beta'-\beta}(1 + T^{2\beta}|||U|||^2). \end{aligned}$$

□

Now we establish a result related with the contraction property of \mathcal{T} :

Lemma 4.10. *Suppose Hypothesis **H** holds. Then there exists $C > 0$ such that for $T \in [0, 1]$ and $U^1 = (u^1, v^1)$, $U^2 = (u^2, v^2) \in W(0, T)$ with $u^1(0) = u_0^1$, $u^2(0) = u_0^2 \in V_\delta$ we have that*

$$|||\mathcal{T}(U^1) - \mathcal{T}(U^2)||| \leq CT^{\beta'-\beta}(1 + T^{2\beta}(|||U^1|||^2 + |||U^2|||^2))(|||U^1 - U^2||| + |u_0^1 - u_0^2|) + T^{\delta-\beta}|u_0^1 - u_0^2|_{V_\delta}.$$

In addition, for the first component \mathcal{T}_1 of the mapping we get

$$|\mathcal{T}_1(U^1)(T) - \mathcal{T}_1(U^2)(T)|_{V_\delta} \leq CT^{\beta'}(1 + T^{2\beta}(|||U^1|||^2 + |||U^2|||^2))(|||U^1 - U^2||| + |u_0^1 - u_0^2|_{V_\delta}) + c|u_0^1 - u_0^2|_{V_\delta}.$$

Proof. Trivially, $\|S(\cdot)u_0^1 - S(\cdot)u_0^2\|_\beta \leq T^{\delta-\beta}|u_0^1 - u_0^2|_{V_\delta}$.

We only give an idea of the proof. Denote $\Delta u = u^1 - u^2$. In particular, in the fractional derivatives containing $G(u)$ in (4.2) and (4.3) we should replace it by $G(u^1) - G(u^2)$ and take into account Lemma 2.2. Now the corresponding estimate is given by

$$\begin{aligned} \|G(u^1(r)) - G(u^2(r))\|_{L_2(V, V_\delta)} & \leq c_{DG}(|u^1(r) - u_0^1 - u^2(r) + u_0^2| + |u_0^1 - u_0^2|) \\ & \leq c_{DG} \left(\sup_{0 \leq q < r \leq T} \frac{|u^1(r) - u^2(r) - (u^1(q) - u^2(q))|}{|r - q|^\beta} T^\beta + |u_0^1 - u_0^2| \right) \\ & = c_{DG}(\|\Delta u\|_\beta T^\beta + |u_0^1 - u_0^2|) \end{aligned}$$

and similar for $DG(u(r))$. We also should use that

$$\begin{aligned} \|DG(u^1(r)) - DG(u^2(r)) - (DG(u^1(q)) - DG(u^2(q)))\|_{L_2(V, V_\delta)} \\ \leq c_{D^2G}\|\Delta u\|_\beta|r - q|^\beta + c_{D^3G}(\|\Delta u\|_\beta T^\beta + |u_0^1 - u_0^2|)(\|u^1\|_\beta + \|u^2\|_\beta)|r - q|^\beta \end{aligned}$$

(see Lemma 2.2) and that

$$\begin{aligned} & \|G(u^1(r)) - G(u^1(q)) - DG(u^1(q))(u^1(r) - u^1(q)) \\ & \quad - (G(u^2(r)) - G(u^2(q)) - DG(u^2(q))(u^2(r) - u^2(q)))\|_{L_2(V, V_\delta)} \\ & \leq c_{D^2G}(\|u^1\|_\beta + \|u^2\|_\beta)\|\Delta u\|_\beta|r - q|^{2\beta} \\ & \quad + c_{D^3G}\|u^2\|_\beta|r - q|^\beta(\|\Delta u\|_\beta T^\beta + |u_0^1 - u_0^2|)(2\|u^1\|_\beta + \|u^2\|_\beta)|r - q|^\beta, \end{aligned}$$

which is true because of Lemma 2.3.

Note that the proof of this result follows in a similar manner as in the previous Lemmata 4.7 and 4.9 by doing the above changes in the integrals representing the equation (4.2) as well as in the integrals related to the area equation (4.3).

To obtain the second statement of this result we would need to use the previous estimates and follow similar steps than in the proof of Lemma 4.7, see Appendix Section. \square

Remark 4.11. *We want to stress that in the previous result we have compared $\mathcal{T}(U^1)$ with $\mathcal{T}(U^2)$ by using the $||| \cdot |||$ -seminorm. As we already mentioned, we could add to the $||| \cdot |||$ -seminorm the V -norm of the initial condition. However, in practice this is not necessary, except when the previous result is used with U^1 and U^2 having different initial conditions, which will happens only in Theorem 4.15 and Lemma 6.6 below. In the latter results in fact we have a sequence of initial conditions which converges and therefore it suffices to consider the seminorm.*

In Theorem 4.15 and the Appendix section we will need to apply the previous lemma when having (u, v) driven by ω and $(u, u \otimes \omega^n)$ driven by ω^n . This is the reason to explain next what happens in this particular situation, which is not included above since the driving noises are different.

Let us indicate for the following lemma the dependence of \mathcal{T} on $u_0 \in V_\delta, \omega, (\omega \otimes_S \omega)$ by $\mathcal{T}(U, \omega, (\omega \otimes_S \omega), u_0)$.

Lemma 4.12. *Suppose Hypothesis **H** holds. Then we have for any $K > 0$ that*

$$\lim_{n \rightarrow \infty} \sup\{|||\mathcal{T}(U, \omega, (\omega \otimes_S \omega), u_0) - \mathcal{T}(U, \omega^n, (\omega^n \otimes_S \omega^n), u_0)||| : |||U||| \leq K, |u_0|_{V_\delta} \leq K\} = 0.$$

Proof. We only sketch the proof. For the first integral on the right hand side of (4.2) we obtain the estimate $2c(1 + K^2)\|\omega^n - \omega\|_{\beta'}$ which tends to zero for $n \rightarrow \infty$ by our Hypothesis **H**. The estimate of the second integral of (4.3) is straightforward thanks to Hypothesis **H**, item (5). Consider finally the first and the third integral of (4.3) which can be written as

$$\begin{aligned} (4.20) \quad & \int_s^t (S(\xi - s) - \text{id})u(s) \otimes_V d(\omega(\xi) - \omega^n(\xi)) \\ & - (-1)^{2\alpha-1} \int_s^t D_{s+}^{2\alpha-1} DG(u(\cdot))[r] D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha}((u \otimes (\omega \otimes_S \omega)(t)) - (u \otimes (\omega^n \otimes_S \omega^n)(t)))[r] dr. \end{aligned}$$

Recall that $(u \otimes (\omega \otimes_S \omega))$ is a function depending on $U = (u, v), \omega, (\omega \otimes_S \omega)$. Let us pick one of the terms we have to estimate, for instance, the following term appearing in $\omega_S(t) \otimes \omega$:

$$\begin{aligned} & \|(\omega^n)_S(\tau, t) \int_s^\tau S(\tau - r) \tilde{E}(\cdot, d\omega^n) - \omega_S(\tau, t) \int_s^\tau S(\tau - r) \tilde{E}(\cdot, d\omega)\|_{L_2(V, V \otimes V)} \\ & \leq c(t - \tau)^{\beta'} (\tau - s)^{\beta'} \|\tilde{E}\|_{L_2(V \otimes V, V_\delta)} (\|\omega - \omega^n\|_{\beta'} (\|\omega\|_{\beta''} + \|\omega\|_{\beta'}) + \|\omega\|_{\beta'} (\|\omega - \omega^n\|_{\beta''} + \|\omega - \omega^n\|_{\beta'})), \end{aligned}$$

which converges to zero. In a similar manner we can estimate the other terms such that we have

$$\|(u \otimes (\omega \otimes_S \omega)(t))(s, q) - (u \otimes (\omega^n \otimes_S \omega^n)(t))(s, q)\|_{L_2(L_2(V \times V, V_\delta), V \otimes V)} \leq C_n |||U||| (q - s)^{\beta+\beta'} (t - s)^{\beta'}.$$

The constant C_n which depends on $\|\omega^n - \omega\|_{\beta'}$ and $\|\omega^n \otimes_S \omega^n\|_{2\beta'}$ and other terms converges to zero for $n \rightarrow \infty$. \square

Theorem 4.13. *Assume U^1, U^2 are two solutions of the system (4.2)-(4.3) in $W(0, 1)$ with initial condition $u_0 \in V_\delta$ for $\delta \in [0, 1]$. Then $U^1 = U^2$.*

Proof. Suppose $U^1 \neq U^2 \in W(0, 1)$. Then there is a ρ such that $|||U^1 - U^2||| = \rho > 0$. By Lemma 6.6 (see Appendix) and Hypothesis **H**, item (5), we can approximate these solutions by sequences $(U_n^1)_{n \in \mathbb{N}}$, $(U_n^2)_{n \in \mathbb{N}}$ having the initial condition u_0 , being $U_n^i = (u_n^i, v_n^i)$ where u_n^i is given by (3.1) and v_n^i has the interpretation of (3.3) driven by smooth ω^n and $(\omega^n \otimes_S \omega^n)$, such that for sufficiently large n we have that $|||U^i - U_n^i||| < \rho/2$ for $i = 1, 2$. However, we have $u_n^1 = u_n^2$ and hence $U_n^1 = U_n^2$ which contradicts $|||U^1 - U^2||| = \rho > 0$. \square

For $0 < a < b$ we consider the *concatenation* of elements of $W(0, a)$ and $W(a, b)$. We have to take into account that the elements of these function spaces consists of a path component and an area component. We have to define the concatenation of the area component in agreement with the Chen equality. Let $U^1 = (u^1, v^1) \in W(0, a)$ such that $u^1(0) \in V_\delta$, for $\delta \in [0, 1]$ and $U^2 = (u^2, v^2) \in W(a, b)$ such that $u^2(a) = u^1(a)$, for $0 \leq a < b \leq 1$. Define $U = (u, v)$ as follows:

$$u(t) = \begin{cases} u^1(t) & : 0 \leq t \leq a \\ u^2(t) & : a \leq t \leq b \end{cases}$$

$$v(s, t) = \begin{cases} v^1(s, t) & : 0 \leq s \leq t \leq a \\ v^2(s, t) & : a \leq s \leq t \leq b \\ (u^1(a) - u^1(s)) \otimes_V (\omega(t) - \omega(a)) + v^1(s, a) + v^2(a, t) & : s \leq a < t. \end{cases}$$

Theorem 4.14. *Suppose Hypothesis **H** holds. In addition, assume that G has its image in $L_2(V, V_\delta)$ with $\delta + \beta' > 1$, $\delta \leq 1$, and that $u_0 \in V_\delta$. Then there exists a unique solution to system (4.2)-(4.3) in $W(0, 1)$.*

Proof. We start presenting a few trivial inequalities. Let C be the common constant such that Lemma 4.7, 4.9 and 4.10 hold. For the following, we have that for any $\rho_0 > 0$ there is a $K(\rho_0) \geq 1$ such that for $K \geq K(\rho_0)$, $i \in \mathbb{N}$

$$(4.21) \quad \rho_0 + \sum_{j=1}^i 2C(Kj)^{-\beta'} = \rho_0 + 2CK^{-\beta'} \sum_{j=1}^i j^{-\beta'} \leq \rho_0 + 2CK^{-\beta'} \frac{1}{1-\beta'} i^{1-\beta'} < (Ki)^{1-\beta'},$$

and

$$(4.22) \quad \begin{aligned} 4C^2((Ki)^{-\beta'-\beta}((Ki)^{\beta-\delta}(Ki)^{1-\beta'} + (Ki)^{\beta-\beta'})) &\leq C'(Ki)^{1-2\beta'-\delta} < 1, \\ C(Ki)^{\beta-\beta'}(1 + 2(Ki)^{-2\beta}(8C^2(Ki)^{2\beta-2\delta}(Ki)^{2-2\beta'} + 8C^2(Ki)^{2\beta-2\beta'})) \\ &\leq C(Ki)^{\beta-\beta'} + C'((Ki)^{\beta-3\beta'-2\delta+2} + (Ki)^{\beta-3\beta'}) < \frac{1}{2}, \\ C(Ki)^{-\beta'} + C(Ki)^{-\beta'-2\beta}(8C^2(Ki)^{2\beta-2\delta}(Ki)^{2-2\beta'} + 8C^2(Ki)^{2\beta-2\beta'}) &< 2C(Ki)^{-\beta'}, \end{aligned}$$

where C' is an appropriate constant independent of i . Note that from (4.21) we also have that $\rho_0 < K^{1-\beta'}$. Moreover, all inequalities in (4.22) are true since we assume that $\beta' + \delta > 1$. The first one is true since we assume that $\delta \leq 1$. Define $|u_0|_{V_\delta} =: \rho_0$, $\Delta T_1 = K^{-1} \leq 1$, $T_1 = T_0 + \Delta T_1$ where $T_0 = 0$. Then, by Lemma 4.7 (i) and Lemma 4.9, we have that

$$|||\mathcal{T}(U)||| \leq C(\Delta T_1^{\delta-\beta} \rho_0 + \Delta T_1^{\beta'-\beta} + \Delta T_1^{\beta'+\beta} |||U|||^2).$$

Hence, to find a ball $B_{W(T_0, T_1)}(0, R_1)$ that will be mapped into itself we calculate the minor root R_1 of

$$(4.23) \quad x = C(\Delta T_1^{\delta-\beta} \rho_0 + \Delta T_1^{\beta'-\beta} + \Delta T_1^{\beta'+\beta} x^2)$$

which is given by

$$\frac{2C(\Delta T_1^{\delta-\beta} \rho_0 + \Delta T_1^{\beta'-\beta})}{1 + \sqrt{1 - 4C^2 \Delta T_1^{\beta'+\beta} (\Delta T_1^{\delta-\beta} \rho_0 + \Delta T_1^{\beta'-\beta})}} < 2C(\Delta T_1^{\delta-\beta} \rho_0 + \Delta T_1^{\beta'-\beta}),$$

see Sohr [28] Page 349. This root is well-defined which follows from (4.21) and the first inequality of (4.22) for $i = 1$, since these conditions in particular imply that

$$(4.24) \quad 1 - 4C^2 \Delta T_1^{\beta'+\beta} (\Delta T_1^{\delta-\beta} \rho_0 + \Delta T_1^{\beta'-\beta}) > 0.$$

Moreover, we obtain from Lemma 4.10 with $u_0^1 = u_0^2$ that \mathcal{T} is a contraction on the ball $B_{W(T_0, T_1)}(0, R_1)$ if

$$C\Delta T_1^{\beta'-\beta}(1 + 2\Delta T_1^{2\beta}(8C^2\Delta T_1^{2\delta-2\beta}\rho_0^2 + 8C^2\Delta T_1^{2\beta'-2\beta})) < \frac{1}{2}$$

which follows from (4.21) and the second inequality of (4.22) for $i = 1$. Then the system (4.2)-(4.3) has a solution U^1 in $B_{W(T_0, T_1)}(0, R_1)$ which is unique by Theorem 4.13.

Furthermore, by Lemma 4.7 (ii) it is known that

$$\begin{aligned} |u(T_1)|_{V_\delta} &\leq \rho_0 + C(\Delta T_1^{\beta'}(1 + \Delta T_1^{2\beta}|||U|||^2)) \\ &\leq \rho_0 + C\Delta T_1^{\beta'} + C\Delta T_1^{\beta'+2\beta}R_1^2 \\ &\leq \rho_0 + C\Delta T_1^{\beta'} + C\Delta T_1^{\beta'+2\beta}(8C^2\Delta T_1^{2\delta-2\beta}\rho_0^2 + 8C^2\Delta T_1^{2\beta'-2\beta}) \\ &< \rho_0 + 2C\Delta T_1^{\beta'} \leq \rho_0 + 2CK^{-\beta'}. \end{aligned}$$

Hence, by using again (4.21), the right hand side of the previous inequality is bounded by $K^{1-\beta'}$.

Suppose now that we have concatenated a solution on $[0, T_{i-1}]$ and that $|u(T_{i-1})|_{V_\delta} < \rho_0 + \sum_{j=1}^{i-1} 2C(Kj)^{-\beta'}$ for $i = 2, 3, \dots$, and $T_{i-1} < 1$. For the fact that this concatenation is a solution we refer to Theorem 4.15 below. Set $T_i = T_{i-1} + \Delta T_i$, $\Delta T_i = (Ki)^{-1}$ if $T_i < 1$, and $T_i = 1$ in other case. By (4.21) we know that $|u(T_{i-1})|_{V_\delta} < (K(i-1))^{1-\beta'}$. Because of (4.22), the Banach fixed point theorem gives us a solution to the system (4.2)-(4.3) in $B_{W(T_{i-1}, T_i)}(0, R_i)$ which is unique, where R_{i-1} is the minor root of (4.23) when replacing ρ_0 by $(K(i-1))^{1-\beta'} < (Ki)^{1-\beta'}$ and ΔT_1 by ΔT_i . Again, by concatenation we obtain a solution on $[0, T_i]$. In addition, we obtain

$$|u(T_i)|_{V_\delta} \leq \rho_0 + \sum_{j=1}^{i-1} 2C(Kj)^{-\beta'} + 2C(Ki)^{-\beta'} = \rho_0 + \sum_{j=1}^i 2C(Kj)^{-\beta'} < (Ki)^{1-\beta'}.$$

Finally, since $\sum_i i^{-1} = \infty$ there is an $i^* \in \mathbb{N}$ such that $T_{i^*} \wedge 1 = 1$, which means that there exists a global solution of (4.2)-(4.3) in $W(0, 1)$ for any $u_0 \in V_\delta$. □

Now we prove the assertion from the last theorem allowing us to concatenate solutions to another solution.

Theorem 4.15. *Assume that Hypothesis **H** holds. Let U^i be the elements from the proof of Theorem 4.14. Then these elements can be concatenated to a solution U in $W(0, 1)$. In particular, this solution satisfies the Chen equality.*

Proof. Let us denote by $\mathcal{T} = \mathcal{T}(U, \omega, (\omega \otimes_S \omega), u_0)$ the right hand side of (4.2), (4.3). In addition, suppose that (u_0^n) converges to u_0 in V_δ . Let $B_{W(T_0, T_1)}(0, R_1)$ be the ball from the proof of Theorem 4.14 such that $\mathcal{T}(U, \omega, (\omega \otimes_S \omega), u_0)$ is a self-mapping and a contraction on this ball with a contraction constant less than $1/2$. We can choose an $R'_1 > R_1$ that $\mathcal{T}(U, \omega, (\omega \otimes_S \omega), u_0)$ is still a contraction with constant less than $1/2$ with respect to $B_{W(T_0, T_1)}(0, R'_1)$. Since the constant C in Lemmata 4.7, 4.9 and 4.10 depends continuously on $u_0, \omega, (\omega \otimes_S \omega)$, for sufficiently large n the mappings $\mathcal{T}(\cdot, \omega^n, (\omega^n \otimes_S \omega^n), u_0^n)$ map $B_{W(T_0, T_1)}(0, R'_1)$, $R'_1 \leq R'_1$ into itself and have a contraction constant less than $1/2$ on these balls. Let $U_1 = (u_1, v_1)$ and $U_1^n = (u_1^n, v_1^n)$ be the fixed points of $\mathcal{T}(\cdot, \omega, (\omega \otimes_S \omega), u_0)$, $\mathcal{T}(\cdot, \omega^n, (\omega^n \otimes_S \omega^n), u_0^n)$. Then

$$\begin{aligned} |||U_1 - U_1^n||| &\leq |||\mathcal{T}(U_1, \omega, (\omega \otimes_S \omega), u_0) - \mathcal{T}(U_1, \omega^n, (\omega^n \otimes_S \omega^n), u_0^n)||| \\ &\quad + |||\mathcal{T}(U_1, \omega^n, (\omega^n \otimes_S \omega^n), u_0^n) - \mathcal{T}(U_1^n, \omega^n, (\omega^n \otimes_S \omega^n), u_0^n)||| \\ &\leq |||\mathcal{T}(U_1, \omega, \omega \otimes_S \omega, u_0) - \mathcal{T}(U_1, \omega^n, (\omega^n \otimes_S \omega^n), u_0^n)||| + \frac{1}{2} |||U_1 - U_1^n||| \end{aligned}$$

such that U_1^n converges to U_1 by Lemmata 4.10 and 4.12.

Applying the Lemma 4.10, with the difference that now we are also considering different driving noises, it holds

$$(4.25) \quad |u_1^n(T_1) - u_1(T_1)|_{V_\delta} \leq c|u_0^n - u_0|_{V_\delta} + cT_1^{\beta'}(1 + T_1^{\beta'}(\|U_1^n\|^2 + \|U_1\|^2))(\|U_1^n - U_1\| + |u_0^n - u_0|_{V_\delta}) \\ + |\mathcal{T}_1(U_1^n, \omega, (\omega \otimes_S \omega), u_0^n) - \mathcal{T}_1(U_1^n, \omega^n, (\omega^n \otimes_S \omega^n), u_0^n)|_{V_\delta}.$$

This inequality implies that $u_1^n(T_1)$ converges to $u_1(T_1)$ in V_δ . The first term on the right hand side is stemming from Lemma 4.10 for the noise path ω . For the convergence of the second expression on the right hand side we note that $\{\|U_1^n\| : n \in \mathbb{N}\}$ is bounded and that the first integral of (4.2) converges when we replace ω by $\omega - \omega^n$ by Hypothesis **H**. Similar we obtain the convergence of the second integral by the convergence of U_1^n to U_1 . Therefore we can repeat the same calculations on $[T_1, T_2]$ and similarly on any of the *finitely* many intervals $[T_i, T_{i+1}]$. Since U_i^n are related to *classical* mild solutions to (1.1) we can concatenate these elements to one element in $W(0, 1)$ where we have to apply Lemma 6.5. These concatenations converge to a solution of (4.2), (4.3) on $W(0, 1)$. \square

5. EXAMPLE: THE WHITE NOISE CASE

In this section we consider only a white noise, i.e. a fractional Brownian motion with Hurst parameter $H = 1/2$. But we note that results from [8] Proposition 3.5 and a generalization of the integration by parts formula would allow us also to consider a fractional Brownian motion with H contained in a region of $(1/3, 1/2]$. However, for brevity we only study here the case $H = 1/2$.

The aim of this section is to give an example such that item (4) and item (5) in Hypothesis **H** hold. At the end of this section we also present two examples of possible non-linear operator G satisfying all assumptions described in Hypothesis **H**.

Let Q be a positive symmetric operator of trace class on V , i.e., $\text{tr}_V Q < \infty$, with positive discrete spectrum $(q_i)_{i \in \mathbb{N}}$ and eigenelements $(f_i)_{i \in \mathbb{N}}$. It is known that then there exists a canonical Wiener process in V given by $(C_0(\mathbb{R}; V), \mathcal{B}(C_0(\mathbb{R}; V)), \mathbb{P})$, where \mathbb{P} is the Wiener measure on $\mathcal{B}(C_0(\mathbb{R}; V))$ determined by Q . The completion of this probability space is denoted by $(\Omega, \mathcal{F}, \mathbb{P})$. Related to the canonical Brownian motion let $(\mathcal{F}_t)_{t \in \mathbb{R}}$ be a right continuous filtration containing all zero sets of \mathcal{F} . This process has for any $\beta' < 1/2$ a β' -Hölder continuous version for which paths will be denoted by ω . Then $\omega_i(t) = q_i^{-\frac{1}{2}}(\omega(t), f_i)_V$ generates an iid-sequence of standard Wiener processes in \mathbb{R} . Nevertheless, for the sake of brevity we assume that the f_i are the same as the e_i .

Let us also denote by ω^n, ω_i^n its piecewise linearizations with respect to the equidistant partition $\{t_i^n\}_{i=0, \dots, n}$ of $[0, 1]$. We then have $\sum_i q_i < \infty$. Recall that for $E \in L_2(V, V_\delta)$

$$E(\omega \otimes_S \omega)(s, t) = \int_s^t \int_s^\xi S(\xi - r) E d\omega(r) \otimes_V \circ d\omega(\xi).$$

Let $\theta_s \omega(\cdot) = \omega(\cdot + s) - \omega(s)$, $s \in \mathbb{R}$, defined on the probability space introduced below, which is again a Wiener process. Then we can choose for the above integral the following version

$$E(\omega \otimes_S \omega)(s, t) = \int_0^{t-s} \int_0^\xi S(\xi - r) E d\theta_s \omega(r) \otimes_V \circ d\theta_s \omega(\xi).$$

Note that it is not important if we consider the Ito- or Stratonovich type of the interior integral, since according to Da Prato and Zabczyk [7] Chapter 5, we have the following interpretation of this integral:

$$\int_0^\xi S(\xi - r) E d\theta_s \omega(r) = E\theta_s \omega(\xi) + \int_0^\xi AS(\xi - r) E\theta_s \omega(r) dr.$$

Now integrating the right hand side of the previous expression with respect to $\otimes_V \circ d\theta_s \omega(\xi)$, performing the transformation $r \rightarrow r - s$, yields

$$E(\omega \otimes_S \omega)(s, t) = E(\omega \otimes \omega)(s, t) + \int_s^t \int_s^\xi AS(\xi - r) E(\omega(r) - \omega(s)) dr \otimes_V \circ d\omega(\xi).$$

We start proving a result related to the first term on the right hand side of the previous expression. We want to stress that in the proof we do not use the exact modulus of continuity of the Brownian motion, as in [17], but its Hölder continuity. In the following we denote the mapping

$$E \rightarrow \int_s^t E(\omega(\xi) - \omega(s)) \otimes_V \circ d\omega(\xi) \quad \text{by} \quad \int_s^t (\omega(\xi) - \omega(s)) \otimes_V \circ d\omega(\xi).$$

Theorem 5.1. *Suppose that Hypothesis **H** holds. Then on a set of full measure*

$$\sup_{0 \leq s < t \leq 1} \frac{\left\| \int_s^t (\omega(\xi) - \omega(s)) \otimes_V \circ d\omega(\xi) \right\|_{L_2(L_2(V, V_\delta), V \otimes V)}}{(t-s)^{2\beta'}} < \infty,$$

and

$$(5.1) \quad \lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq 1} \frac{\left\| \int_s^t (\omega(\xi) - \omega(s)) \otimes_V \circ d\omega(\xi) - \int_s^t (\omega^n(\xi) - \omega^n(s)) \otimes_V \circ d\omega^n(\xi) \right\|_{L_2(L_2(V, V_\delta), V \otimes V)}}{(t-s)^{2\beta'}} = 0.$$

Proof. Consider the orthonormal basis $(E_{ki})_{k,i \in \mathbb{N}}$ of $L_2(V, V_\delta)$ introduced in section 3 given by

$$E_{ki}e_l = \begin{cases} 0 & : i \neq l \\ \frac{e_k}{\lambda_k^\delta} & : i = l. \end{cases}$$

In addition, $(e_l \otimes_V e_j)_{l,j \in \mathbb{N}}$ is an orthonormal basis in $V \otimes V$. The first assertion of this theorem follows simply from estimating

$$\begin{aligned} \left\| \int_s^t (\omega(\xi) - \omega(s)) \otimes_V \circ d\omega(\xi) \right\|_{L_2(L_2(V, V_\delta), V \otimes V)}^2 &= \sum_{i,k} \sum_{l,j} \left(\int_s^t E_{ki}(\omega(\xi) - \omega(s)) \otimes_V \circ d\omega(\xi), e_l \otimes_V e_j \right)_{V \otimes V}^2 \\ &\leq \sum_k \lambda_k^{-2\delta} \sum_{k,j} q_k q_j \left(\int_s^t (\omega_k(\xi) - \omega_k(s)) d\omega_j(\xi) \right)^2. \end{aligned}$$

Moreover, we obtain a similar estimate than above for the numerator of (5.1). Let us denote

$$A_{i,j}^n(s, t) := \int_s^t (\omega_i(\xi) - \omega_i(s)) d\omega_j(\xi) - \int_s^t (\omega_i^n(\xi) - \omega_i^n(s)) d\omega_j^n(\xi).$$

By symmetry we can assume $i \leq j$. In fact we assume that $i < j$ since the case $i = j$ is easier, see a comment at the end of the proof. Note that even though our tensors are defined in the set $\{(s, t) \in [0, 1]^2 : 0 \leq s \leq t \leq 1\}$, since the Lévy area $A_{i,j}^n$ is symmetric they can be extended to the set $[0, 1]^2$.

To estimate $A_{i,j}^n(s, t)$ we apply the Garsia–Rodemich–Rumsley Lemma (Lemma 3.4) given by Deya et al. [8], which claims that for $p \geq 1$ there exists $K_{\beta', p}$ such that

$$(5.2) \quad \|A_{i,j}^n\|_{2\beta'} \leq K_{\beta', p} (R_{n,p}^{i,j} + \|\omega_i - \omega_i^n\|_{\beta'} \|\omega_j\|_{\beta'} + \|\omega_j - \omega_j^n\|_{\beta'} \|\omega_i^n\|_{\beta'}),$$

where

$$R_{n,p}^{i,j} := \left(\int_0^1 \int_0^1 \frac{|A_{i,j}^n(s, t)|^{2p}}{|t-s|^{4\beta'p+2}} ds dt \right)^{1/(2p)}.$$

In particular, we know from the proof of Lemma 3.7 in [8] that

$$\mathbb{E}(R_{n,p}^{i,j})^{2p} \leq n^{-4p(1/2-\beta'')} < \infty,$$

for $\beta' < \beta'' < 1/2$ and p chosen sufficiently large such that $4p(1/2 - \beta'') > 1$. Then

$$\mathbb{P}(\sum_{ij} q_i q_j (R_{p,n}^{i,j})^2 > o_n^2) \leq (\text{tr} Q)^{2(p-1)} \sum_{ij} q_i q_j \mathbb{E}(R_{n,p}^{i,j})^{2p} o_n^{-2p} \leq C n^{-4p(1/2-\beta'')} o_n^{-2p}.$$

For an appropriate sequence $(o_n)_{n \in \mathbb{N}}$ with limit zero, the right hand side has a finite sum. Then by the Borel-Cantelli Lemma, $(\sum_{ij} q_i q_j (R_{p,n}^{ij})^2)_{n \in \mathbb{N}}$ tends to zero almost surely. In a similar manner we obtain the convergence of the tail terms in (5.2). However instead of using the exact modulus of continuity of the Brownian motion we apply that, for $\beta' < \beta'' < 1/2$,

$$(5.3) \quad \|\omega_i - \omega_i^n\|_{\beta'} \leq G_{\beta''}(i, \omega) n^{\beta' - \beta''}, \quad \|\omega_i\|_{\beta'} \leq G_{\beta''}(i, \omega), \quad \|\omega_i^n\|_{\beta'} \leq G_{\beta''}(i, \omega)$$

where $G_{\beta''}(i, \omega) \geq \|\omega_i\|_{\beta''}$ and $G_{\beta''}(n, \omega) \in L_p(\Omega)$ for $p \in \mathbb{N}$ are iid random variables, see Kunita [20] Theorem 1.4.1. We then have

$$\begin{aligned} \mathbb{P}(\sum_{ij} q_i q_j \|\omega_i^n - \omega_i\|_{\beta'}^2 \|\omega_j\|_{\beta'}^2 > o_n^2) &\leq (\text{tr} Q)^{2(p-1)} \sum_{ij} q_i q_j (\mathbb{E} G_{\beta''}(i, \omega)^{4p})^{\frac{1}{2}} (\mathbb{E} G_{\beta''}(j, \omega)^{4p})^{\frac{1}{2}} n^{-2p(\beta'' - \beta')} o_n^{-2p} \\ &\leq C n^{-2p(\beta'' - \beta')} o_n^{-2p}. \end{aligned}$$

For p chosen sufficiently large and an appropriate zero-sequence $(o_n)_{n \in \mathbb{N}}$ we obtain the almost sure convergence of $(\sum_{ij} q_i q_j \|\omega_i^n - \omega_i\|_{\beta'}^2 \|\omega_j\|_{\beta'}^2)_{n \in \mathbb{N}}$. Similarly we can treat the last term of (5.2), that is, $(\sum_{ij} q_i q_j \|\omega_i^n - \omega_i\|_{\beta'}^2 \|\omega_j^n\|_{\beta'}^2)_{n \in \mathbb{N}}$. Finally, note that

$$A_{i,i}^n(s, t) = \frac{1}{2}(\omega_i(t) - \omega_i(s))^2 - \frac{1}{2}(\omega_i^n(t) - \omega_i^n(s))^2$$

and thanks to (5.3) we can obtain $\|A_{i,i}^n\|_{2\beta'} \leq G_{\beta''}^2(i, \omega) n^{\beta' - \beta''}$, which completes the proof. \square

Lemma 5.2. *Suppose that Hypothesis **H** holds and that for some $\gamma > 1/2$*

$$(5.4) \quad \sum_i \lambda_i^{2\gamma - 2\delta} < \infty.$$

Then

$$\int_s^t AS(t-r)E(\omega(r) - \omega(s))dr \in L_2(L_2(V, V_\delta), V)$$

defines an element of $C_\gamma([s, 1]; L_2(L_2(V, V_\delta), V))$ for $s \in [0, 1)$ where the norm is independent of $s \in [0, 1]$.

Proof. We know that $\int_0^t S(-r)\omega(r)dr \in C_\gamma([0, 1]; V_{1-\gamma})$ where the norm is bounded by $c\|\omega\|_{\beta'}$, see for instance Bensoussan and Frehse [1], Corollary 2.1. Note that this estimate remains true if we replace the interval $[0, 1]$ there by $[s, 1]$. Moreover,

$$\begin{aligned} &\sup_{s \leq \tilde{t} < t \leq 1} \frac{\left(\sum_{ij} \left| \int_s^t AS(t-r)E_{ij}(\omega(r) - \omega(s))dr - \int_s^{\tilde{t}} AS(\tilde{t}-r)E_{ij}(\omega(r) - \omega(s))dr \right|^2 \right)^{\frac{1}{2}}}{|t - \tilde{t}|^\gamma} \\ &= \sup_{s \leq \tilde{t} < t \leq 1} \frac{\left(\sum_{ij} \left| \int_s^t A^{1-\gamma} S(t-r) A^\gamma E_{ij}(\omega(r) - \omega(s))dr - \int_s^{\tilde{t}} A^{1-\gamma} S(\tilde{t}-r) A^\gamma E_{ij}(\omega(r) - \omega(s))dr \right|^2 \right)^{\frac{1}{2}}}{|t - \tilde{t}|^\gamma} \\ &\leq c \left(\sum_i \lambda_i^{2\gamma - 2\delta} \right)^{\frac{1}{2}} \left(\sum_j q_j \|\omega_j\|_{\beta'}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Similar arguments given in (5.3) allows to conclude that the last sum under the square root is finite almost surely. \square

Define

$$(5.5) \quad F_s(\omega, t, E) = \int_s^t AS(t-r)E(\omega(r) - \omega(s))dr.$$

We note that because $\gamma > 1/2$, the element $F_s(E)$ given by (5.5) has square variation zero: for any sequence of partitions $\{t_i^n\}$ of $[0, 1]$ with meshsize P_n going to zero we have

$$\lim_{P_n \rightarrow 0} \sum_{i=1}^n \|(F_s(\omega, t_{i-1}^n) - F_s(\omega, t_i^n))\|_{L_2(L_2(V, V_\delta), V)}^2 = 0.$$

In addition, Stratonovich integrals can be expressed by pathwise integrals in terms of fractional derivatives, see [17].

Theorem 5.3. *Let F_s given by (5.5). Assuming Hypothesis **H** and (5.4), on a set of full measure there is $c > 0$ such that for $0 \leq s < t \leq 1$*

$$\left\| \int_s^t F_s(\omega, r) \otimes_V \circ d\omega(r) \right\|_{L_2(L_2(V, V_\delta), V \otimes V)} \leq c(t-s)^{2\beta'},$$

and

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s < t \leq 1} \frac{\left\| \int_s^t F_s(\omega, r) \otimes_V \circ d\omega(r) - \int_s^t F_s(\omega^n, r) \otimes_V d\omega^n(r) \right\|_{L_2(L_2(V, V_\delta), V \otimes V)}}{(t-s)^{2\beta'}} = 0.$$

Proof. We only prove the second relationship since the first one follows similarly. We have

$$\begin{aligned} & \left\| \int_s^t F_s(\omega, r) \otimes_V \circ d\omega(r) - \int_s^t F_s(\omega^n, r) \otimes_V d\omega^n(r) \right\|_{L_2(L_2(V, V_\delta), V \otimes V)} \\ & \leq \left\| \int_s^t F_s(\omega - \omega^n, r) \otimes_V d\omega(r) \right\|_{L_2(L_2(V, V_\delta), V \otimes V)} + \left\| \int_s^t F_s(\omega^n, r) \otimes_V (\circ d\omega - d\omega^n)(r) \right\|_{L_2(L_2(V, V_\delta), V \otimes V)}. \end{aligned}$$

Now for $\gamma > 1/2$ choose $\beta' < 1/2$ such that $\gamma + \beta' > 1$. Thus, we can find an α satisfying $1 - \beta' < \alpha < \gamma$. Then we have for the first integral of the right hand side

$$\left\| \int_s^t F_s(\omega - \omega^n, r, E_{ij}) \otimes_V \circ d\omega(r) \right\| = \left\| \int_s^t D_{s+}^\alpha F_s(\omega - \omega^n, \cdot, E_{ij})[r] \otimes_V D_{t-}^{1-\alpha} \omega_{t-}[r] dr \right\|.$$

Note that, as a consequence of the proof of Lemma 5.2, since $E \mapsto F_s(s, t, E)$ is linear,

$$\sum_{ij} |D_{s+}^\alpha F_s(\omega - \omega^n, \cdot, E_{ij})[r]| \leq c \|\omega - \omega^n\|_{\beta'} (r-s)^{\gamma-\alpha}$$

while $|D_{t-}^{1-\alpha} \omega_{t-}[r]| \leq c \|\omega\|_{\beta'} (t-r)^{\alpha+\beta'-1}$. Lemma 6.1 gives the desired estimate. The second term can be estimated similarly. \square

Let us consider two examples for operators G satisfying the assumptions of Theorem 4.14:

We start by considering some lattice system. Let $V = l_2$ be the space of square additive sequences with values in \mathbb{R} . In addition, let A be a negative symmetric operator defined on $D(-A) \subset l_2$ with compact inverse. In particular, we can assume that $-A$ has a discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_i \leq \dots \rightarrow \infty$ where the associated eigenelements $(e_i)_{i \in \mathbb{N}}$ form a complete orthonormal system in l_2 . The spaces $D((-A)^\nu)$ are then defined by

$$\{u = (u_i)_{i \in \mathbb{N}} \in l_2 : \sum_i \lambda_i^{2\nu} u_i^2 < \infty\}.$$

For $\beta' < 1/2$ we take δ such that $\delta + \beta' > 1$. We then consider the space V_δ for which we assume that for some $\gamma > 1/2$ the condition $\sum_i \lambda_i^{2\gamma-2\delta} < \infty$ holds, and hence (5.4) is satisfied.

Consider a sequence of functions $(g_{ij})_{i,j \in \mathbb{N}}$, with $g_{ij} : V \rightarrow \mathbb{R}$, and define $G(u)$ for $u \in V$ by

$$G(u)v = \left(\sum_j g_{ij}(u)v_j \right)_{i \in \mathbb{N}} \quad \text{for all } v \in V.$$

We assume that

$$\|G(u)\|_{L_2(V, V_\delta)}^2 = \sum_j |G(u)e_j|_{V_\delta}^2 = \sum_j \left(\sum_i \lambda_i^{2\delta} (G(u)e_j)_i^2 \right) = \sum_{i,j} \lambda_i^{2\delta} g_{ij}^2(u) \leq c$$

uniformly with respect to $u \in V$.

In addition, assume that g_{ij} are four times differentiable and their derivatives are uniformly bounded in the following way

$$\begin{aligned} |Dg_{ij}(u)(e_k)| &\leq c_{g,1}^{ijk}, \quad |D^2g_{ij}(u)(e_k, h_1)| \leq c_{g,2}^{ijk}|h|, \quad |D^3g_{ij}(u)(e_k, h_1, h_2)| \leq c_{g,3}^{ijk}|h_1||h_2|, \\ |D^4g_{ij}(u)(e_k, h_1, h_2, h_3)| &\leq c_{g,4}^{ijk}|h_1||h_2||h_3| \quad \text{for any } u \in V, \end{aligned}$$

and these bounds satisfy

$$\sum_{ijk} \lambda_i^{2\delta} (c_{g,1}^{ijk})^2 < \infty, \quad \sum_{ijk} \lambda_i^{2\delta} (c_{g,2}^{ijk})^2 < \infty, \quad \sum_{ijk} \lambda_i^{2\delta} (c_{g,3}^{ijk})^2 < \infty, \quad \sum_{ijk} \lambda_i^{2\delta} (c_{g,4}^{ijk})^2 < \infty.$$

To see for instance that DG exists, note that by Taylor expansion

$$|g_{ij}(u+h) - g_{ij}(u) - Dg_{ij}(u)(h)|^2 \leq \frac{1}{2} |D^2g_{ij}(u+\eta h)(h, h)|^2 \leq (c_{g,2}^{ijk})^2 |h|^2$$

where $u, h \in V$ and $\eta \in [0, 1]$. In particular, we also note that

$$\sum_{j,k} \left| DG(u)(e_k, e_j) \right|_{V_\delta}^2 \leq \sum_{ijk} \lambda_i^{2\delta} (c_{g,1}^{ijk})^2 =: c_{DG}^2 < \infty.$$

This condition ensures the Lipschitz continuity of G as well as the Hilbert-Schmidt property of DG . Similarly, we obtain that DG is also Lipschitz with respect to the Hilbert-Schmidt norm. We also obtain the existence of the second and third derivative. By the choice of β' and δ , the conditions on G in Hypothesis **H** hold.

Now, we consider the second example. Let us assume that A is generated by the Laplacian on $\mathcal{O} = (0, 1)$ with homogenous Dirichlet boundary condition. A generates a semigroup on $L_2(\mathcal{O})$ with domain $D(-A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. Let $\rho \in (1/4, 1)$, then $V := D((-A)^\rho)$ consists of the Slobodetski spaces $H^{2\rho}(\mathcal{O})$ satisfying the homogeneous boundary conditions, see [7], Page 401. In particular, the continuous embedding $V \subset L^5(\mathcal{O})$ holds when $2\rho \geq 3/10$. In what follows we consider the restriction of the semigroup to V . Note that the inequalities (2.1) and (2.2) continue being true, and that $(\lambda_i^{-\rho} e_i)_{i \in \mathbb{N}}$ is a orthonormal basis of V where $(\lambda_i)_{i \in \mathbb{N}}$ is the spectrum of A and $(e_i)_{i \in \mathbb{N}}$ are the associated eigenelements which are uniformly bounded in $L_\infty(\mathcal{O})$. The asymptotical behavior of the spectrum is given by $\lambda_i \sim i^2$.

Now for $\beta' < 1/2$, $\rho \in (1/4, 1)$ we can choose δ such that $\beta' + \delta > 1$, $\rho + \delta \leq 1$, and thus $D((-A)) \subset D((-A)^{\rho+\delta}) =: V_\delta$. Since $\delta + \rho > 3/4$ we can find an $\gamma > 1/2$ such that $\sum_i \lambda_i^{2\gamma-2\delta-2\rho} < \infty$ holds, and thus (5.4) is satisfied.

Let g be a four times continuously differentiable function on $\bar{\mathcal{O}} \times \mathbb{R}$ which is zero on $\{0, 1\} \times \mathbb{R}$, such that all the corresponding derivatives (g itself included) are bounded. Define

$$G(u)(v)[x] = \int_{\mathcal{O}} g(x, u(y))v(y)dy \quad \text{for } u, v \in V.$$

Following Kantorowitsch and Akilow [19] Section XVII.3 it is not hard to prove that G is three times differentiable where the derivatives are given by

$$\begin{aligned} DG(u)(v, h_1)[x] &= \int_{\mathcal{O}} D_2g(x, u)v(y)h_1(y)dy, \\ D^2G(u)(v, h_1, h_2)[x] &= \int_{\mathcal{O}} D_2^2g(x, u)v(y)h_1(y)h_2(y)dy, \\ D^3G(u)(v, h_1, h_2, h_3)[x] &= \int_{\mathcal{O}} D_2^3g(x, u)v(y)h_1(y)h_2(y)h_3(y)dy, \end{aligned}$$

for $v, h_1, h_2, h_3 \in V$. We have that $G(u)(v), DG(u)(v, h_1), D^2G(u)(v, h_1, h_2), D^3G(u)(v, h_1, h_2, h_3) \in D(-A) \subset V_\delta$. Let us check, for instance, that $D^3G(u)(v, h_1, h_2, h_3) \in D(-A) \subset V_\delta$. By the continuous embedding theorem we have that

$$\begin{aligned} & \int_{\mathcal{O}} \left| \int_{\mathcal{O}} \Delta_x D_2^2 g(x, u(y) + h_3(y)) v(y) h_1(y) h_2(y) - \Delta_x D_2^2 g(x, u(y)) v(y) h_1(y) h_2(y) \right. \\ & \quad \left. - D_2 \Delta_x D_2^2 g(x, u(y)) v(y) h_1(y) h_2(y) h_3(y)^2 dy \right|^2 dx \leq c \left(\int_{\mathcal{O}} |v(y) h_1(y) h_2(y) h_3^2(y)| dy \right)^2 \\ & \leq c |v|_{L^5}^2 |h_1|_{L^5}^2 |h_2|_{L^5}^2 |h_3|_{L^5}^4 \leq c' |v|^2 |h_1|^2 |h_2|^2 |h_3|^4 \end{aligned}$$

where c is a uniform bound for $|\Delta_x D_2^4 g(x, u)|^2 |{\mathcal{O}}|$. Furthermore, $G(u)(v)[x], \dots, D^3G(u)(v, h_1, h_2, h_3)[x]$ are zero for $x \in \{0, 1\}$.

The Hilbert-Schmidt property of $DG(u)$ follows by

$$\sum_{i,j} \int_{\mathcal{O}} \left(\int_{\mathcal{O}} |\Delta_x D_2 g(x, u(y)) \lambda_i^{-\rho} e_i(y) \lambda_j^{-\rho} e_j(y) dy \right)^2 dx < c \left(\sum_i \lambda_i^{-2\rho} \right)^2 < \infty,$$

due to the boundedness of $\Delta_x D_2 g$. In the same manner we obtain that the other derivatives are Hilbert-Schmidt operators.

These estimates allow us to apply Theorem 4.14.

6. APPENDIX: A PRIORI ESTIMATES

We start this section by considering a *modified Beta-function*.

Lemma 6.1. *The integral*

$$B_\nu^\eta(a, b) := \int_a^b (b-q)^\eta (q-a)^\nu dq = c(b-a)^{\nu+\eta+1}, \quad c = c(\nu, \eta),$$

for $a < b$ and $\nu, \eta > -1$. In addition,

$$\int_a^b B_\nu^\eta(a, r) (r-a)^\xi (b-r)^\mu dr = c(b-a)^{\nu+\eta+\mu+\xi+2}, \quad c = c(\nu, \eta, \xi, \mu),$$

if $\nu + \eta + \xi > -2$ and $\nu, \eta, \mu > -1$.

The proofs of these equalities are standard from the definition of the Beta-function.

When $\nu = 0$, we denote the corresponding modified Beta-function just by $B^\eta(a, b)$.

We now give some properties of the fractional derivatives $D_{s+}^{2\alpha-1}$ and \hat{D}_{s+}^α . We shall omit their proofs since they follow straightforwardly.

In the following \hat{V} represents an abstract Hilbert space that will be determined in the proof of Lemma 4.7 below.

Lemma 6.2. *Let $[0, T] \ni r \mapsto Q(r) \in L(V)$ be a continuous operator satisfying*

$$\|Q(r) - Q(q)\|_{L(V)} \leq c_Q (r-q)^{\beta_1} \quad \text{for } r > q, \quad -2\alpha + \beta_1 > -1.$$

We also suppose that, for $0 \leq s \leq r \leq T$,

$$\sup_{q \in [s, r]} \|Q(q)\|_{L(V)} \leq c'_Q.$$

In addition, for $\delta \in [0, 1]$, let $V \ni x \mapsto R(x) \in L_2(\hat{V}, V_\delta)$ be a continuously differentiable function bounded by c_R . The first derivative of R is supposed to be bounded by c_{DR} . Then, for $0 \leq s < r \leq T$ and $-2\alpha + \beta > -1$, for $u \in C_\beta([0, T]; V)$ we have

$$|D_{s+}^{2\alpha-1}(Q(\cdot)R(u(\cdot)))[r]| \leq c \left(\frac{c'_Q c_R}{(r-s)^{2\alpha-1}} + c_Q c_R B^{-2\alpha+\beta_1}(s, r) + c'_Q c_{DR} B^{-2\alpha+\beta}(s, r) \|u\|_\beta \right).$$

Lemma 6.3. *Suppose that $2\beta > \alpha$. Let $Q(\cdot)$ be given in Lemma 6.2 such that $\beta_1 > \alpha$ and let R be mapping from V to $L_2(\hat{V}, V_\delta)$ such that*

$$\|R(x) - R(y) - DR(y)(x - y)\|_{L_2(\hat{V}, V_\delta)} \leq c_{D^2R}|x - y|^2 \quad \text{for } x, y \in V.$$

Then, for $0 \leq s < r \leq T$ and $u \in C_\beta([0, T]; V)$ it holds

$$\begin{aligned} |\hat{D}_{s+}^\alpha(Q(\cdot)R(u(\cdot)))[r]| &\leq c \left(\frac{c'_Q \sup_{p \in [0, T]} \|R(u(p))\|_{L_2(\hat{V}, V_\delta)}}{(r - s)^\alpha} \right. \\ &\quad \left. + c_Q B^{-\alpha-1+\beta_1}(s, r) \sup_{p \in [0, T]} \|R(u(p))\|_{L_2(\hat{V}, V_\delta)} + c'_Q c_{D^2R} B^{-\alpha-1+2\beta}(s, r) \|u\|_\beta^2 \right). \end{aligned}$$

Remark 6.4. *We notice that, in Lemma 6.2 and Lemma 6.3, c_Q and c'_Q denote expressions related with the operator Q but have not to be necessarily constants. Moreover, c_Q and c'_Q can depend on parameters.*

We also want to point out that, in the two previous lemmata, we could have considered Q such that $r \in [0, T] \mapsto Q(r) \in L(V_\delta, V)$. However, this assumption would give us no significant improvements to the estimates in Lemma 4.7, thus for the sake of easier presentation, we have assumed $r \in [0, T] \mapsto Q(r) \in L(V)$.

Next, we prove the Lemma 4.7.

Proof. We start assuming that $u_0 \in V_\delta$ for $\delta \geq \beta$.

By (2.1) and (2.2), for $0 < s < t \leq T$ we have that

$$|(S(t) - S(s))u_0| \leq C s^{\delta-\beta} (t - s)^\beta |u_0|_{V_\delta},$$

and then $\|S(\cdot)u_0\|_\beta \leq C|u_0|_{V_\delta} T^{\delta-\beta}$.

Now, in order to complete the proof of (4.19), we are going to estimate the V -norm of the following expression:

$$\begin{aligned} &\int_0^t S(t-r)G(u(r))d\omega(r) - \int_0^s S(s-r)G(u(r))d\omega(r) \\ &= \int_s^t S(t-r)G(u(r))d\omega(r) + \int_0^s (S(t-r) - S(s-r))G(u(r))d\omega(r) \\ &=: A_{11}(s, t) + A_{12}(s, t) + A_{21}(0, s) + A_{22}(0, s) \end{aligned}$$

where $0 < s < t \leq T$, $U = (u, v) \in W(0, T)$ and

$$\begin{aligned} A_{11}(s, t) &= (-1)^\alpha \int_s^t \hat{D}_{s+}^\alpha (S(t-\cdot)G(u(\cdot)))[r] D_{t-}^{1-\alpha} \omega_{t-}[r] dr, \\ A_{12}(s, t) &= -(-1)^{2\alpha-1} \int_s^t D_{s+}^{2\alpha-1} (S(t-\cdot)DG(u(\cdot)))[r] D_{t-}^{1-\alpha} \mathcal{D}_{t-}^{1-\alpha} v(\cdot, t)[r] dr, \\ A_{21}(0, s) &= (-1)^\alpha \int_0^s \hat{D}_{0+}^\alpha ((S(t-\cdot) - S(s-\cdot))G(u(\cdot)))[r] D_{s-}^{1-\alpha} \omega_{s-}[r] dr, \\ A_{22}(0, s) &= -(-1)^{2\alpha-1} \int_0^s D_{0+}^{2\alpha-1} ((S(t-\cdot) - S(s-\cdot))DG(u(\cdot)))[r] D_{s-}^{1-\alpha} \mathcal{D}_{s-}^{1-\alpha} v(\cdot, t)[r] dr. \end{aligned}$$

To do that, we use Lemma 6.1 together with Lemma 6.2 or Lemma 6.3, depending on in the integrand whether the fractional derivative or the compensated fractional derivative appear. As examples, let us show here how to estimate A_{12} and A_{21} (for the rest of cases, the reader could look on the different values of parameters which are on the table 1 below).

For the term A_{12} we make the following identification in Lemma 6.2:

$$\hat{V} = V \otimes V, \quad R = DG, \quad c_R = c_{DG}, \quad c_{DR} = c_{D^2G}, \quad \beta_1 = 2\beta, \quad Q(\cdot) = S(t - \cdot), \quad c_Q = (t - r)^{-2\beta}, \quad c'_Q = c.$$

Therefore, using (4.13) and applying then Lemma 6.1, with $a = s$ and $b = t$, it holds

$$\begin{aligned} |A_{12}(s, t)| &\leq c \int_s^t \left(\frac{1}{(r - s)^{2\alpha-1}} + (t - r)^{-\beta} B^{-2\alpha+\beta}(s, r) \right. \\ &\quad \left. + B^{-2\alpha+\beta}(s, r) \|U\| \right) \|U\| (t - r)^{2\alpha+\beta+\beta'-2} dr, \end{aligned}$$

where we have an β' such that $-3\beta + \alpha + \beta' > 0$. Hence,

$$|A_{12}(s, t)| \leq C(t-s)^{\beta'+\beta}(1 + (t-s)^\beta |||U|||^2).$$

For the term A_{21} we make the following identification in Lemma 6.3:

$$\begin{aligned} \hat{V} &= V, \quad R = G, \quad c_{D^2R} = c_{D^2G}, \quad \beta_1 = 2\beta, \quad \sup_{p \in [0, T]} \|R(u(p))\|_{L_2(V, V_\delta)} \leq c_G, \\ Q(\cdot) &= (S(t - \cdot) - S(s - \cdot)), \quad c_Q = \frac{(t-s)^\beta}{(s-r)^{3\beta}}, \quad c'_Q = \frac{(t-s)^\beta}{(s-r)^\beta}. \end{aligned}$$

Therefore, using (4.7) and applying then Lemma 6.1 with $a = 0$ and $b = s$, we have

$$\begin{aligned} |A_{21}(0, s)| &\leq c \int_0^s \left(\frac{(t-s)^\beta}{(s-r)^{\beta} r^\alpha} + \frac{(t-s)^\beta B^{-\alpha-1+2\beta}(0, r)}{(s-r)^{3\beta}} \right. \\ &\quad \left. + \frac{(t-s)^\beta B^{-\alpha-1+2\beta}(0, r) |||U|||^2}{(s-r)^\beta} \right) (s-r)^{\alpha+\beta'-1} dr. \end{aligned}$$

Hence,

$$|A_{21}(0, s)| \leq Ct^{\beta'}(1 + (t-s)^{2\beta} |||U|||^2).$$

Therefore, the previous estimates imply

$$\left\| \int_0^\cdot S(\cdot - r)G(u(r))d\omega \right\|_\beta \leq CT^{\beta'-\beta}(1 + T^{2\beta} |||U|||^2),$$

and thus (4.19) is proved.

	a	b	β_1	c_Q	c'_Q	$\sup R$	c_R	c_{DR}	ξ
A_{11}	s	t	2β	$(t-r)^{-2\beta}$	c	c_G			
A_{12}	s	t	2β	$(t-r)^{-2\beta}$	c		c_{DG}	c_{D^2G}	$-\beta$
A_{21}	0	s	2β	$\frac{(t-s)^\beta}{(s-r)^{3\beta}}$	$\frac{(t-s)^\beta}{(s-r)^\beta}$	c_G			
A_{22}	0	s	2β	$\frac{(t-s)^\beta}{(s-r)^{3\beta}}$	$\frac{(t-s)^\beta}{(s-r)^\beta}$		c_{DG}	c_{D^2G}	$-\beta$

TABLE 1. Values of the different parameters appearing in the estimates of A_{ij} , for $i, j = 1, 2$.

(ii) We obtain clearly that $|S(T)u_0|_{V_\delta} \leq |u_0|_{V_\delta}$. It is interesting to emphasize here that the constant in the previous estimate is just 1, which is of importance in the proof of Theorem 4.14.

Moreover, since the semigroup and the operator A commute, we can write

$$(-A)^\delta \int_0^t S(t-r)G(u(r))d\omega(r) = \int_0^t S(t-r)(-A)^\delta G(u(r))d\omega(r).$$

In this point, since G takes values in $L_2(V, V_\delta)$ we have that $(-A)^\delta G(\cdot) \in L_2(V)$, and $\|G(u(\cdot))\|_{L_2(V, V_\delta)} = \|(-A)^\delta G(u(\cdot))\|_{L_2(V)}$. Therefore, we can use the properties of Lemma 2.2 and the calculations in part (i) of this proof to conclude (ii). \square

In the following technical lemmata we collect some properties that are needed for the global existence of a solution of (4.2), (4.3).

Lemma 6.5. *Let $U = (u, v) \in W(0, 1)$ be a solution of (4.2), (4.3) for a smooth ω , then $(u \otimes \omega)$ satisfies the Chen equality.*

Proof. For the sake of simplicity, we assume that ω is an almost everywhere differentiable V -valued path.

(i) Examining carefully the proof of Lemma 4.7 we can check that the constant C appearing in that proof depends continuously on $\|\omega\|_{\beta'}$. Doing the same with the proof of Lemma 4.9, we see that the corresponding constant C depends continuously on $\|(\omega \otimes_S \omega)\|_{2\beta'}$ and also on $\|(u \otimes (\omega \otimes_S \omega)(t))\|_{\beta+2\beta'}$, hence the result follows.

(ii) A straightforward calculation shows

$$u(\xi) - u(s) = \int_s^\xi S(\xi - r)G(u(r))d\omega(r) + S(\xi - s)u(s) - u(s),$$

and therefore

$$\begin{aligned} (u \otimes \omega)(s, \tau) &= \int_s^\tau \left(\int_s^\xi S(\xi - r)G(u(r))d\omega(r) + S(\xi - s)u(s) - u(s) \right) \otimes_V d\omega(\xi) \\ (u \otimes \omega)(\tau, t) &= \int_\tau^t \left(\int_\tau^\xi S(\xi - r)G(u(r))d\omega(r) + S(\xi - \tau)u(\tau) - u(\tau) \right) \otimes_V d\omega(\xi) \\ (u \otimes \omega)(s, t) &= \int_s^t \left(\int_s^\xi S(\xi - r)G(u(r))d\omega(r) + S(\xi - s)u(s) - u(s) \right) \otimes_V d\omega(\xi) \end{aligned}$$

We note that for $\xi \in (\tau, t)$, we have

$$S(\xi - \tau)u(\tau) = S(\xi - \tau)S(\tau - s)u(s) + \int_s^\tau S(\xi - \tau)S(\tau - r)G(u(r))d\omega(r),$$

and therefore

$$\begin{aligned} (6.1) \quad & \int_s^\tau S(\xi - s)u(s) \otimes_V d\omega(\xi) + \int_\tau^t S(\xi - \tau)u(\tau) \otimes_V d\omega(\xi) \\ &= \int_s^t S(\xi - s)u(s) \otimes_V d\omega(\xi) + \int_\tau^t \int_s^\tau S(\xi - r)G(u(r))d\omega(r) \otimes_V d\omega(\xi). \end{aligned}$$

Moreover, the rectangular term in the Chen equality can be written as

$$\begin{aligned} (u(\tau) - u(s)) \otimes_V (\omega(t) - \omega(\tau)) &= u(\tau) \otimes_V (\omega(t) - \omega(\tau)) - u(s) \otimes_V (\omega(t) - \omega(s)) \\ &\quad - u(s) \otimes_V (\omega(s) - \omega(\tau)). \end{aligned}$$

Thus, combining the previous equality with (6.1), we obtain the Chen equality for smooth ω . \square

Lemma 6.6. *Suppose Hypothesis **H** holds. Let $U \in W(0, 1)$ be a solution of (4.2), (4.3) with initial condition $u_0 \in V_\delta$ and let U_n be a solution of (3.1) having the same initial condition which can be interpreted in the sense of (4.2), (4.3), see Section 3. Then on $W(0, 1)$*

$$\lim_{n \rightarrow \infty} |||U - U_n||| = 0.$$

Proof. The proof is quite similar to the main Theorem 4.14. Because we assume *a priori* that there exists a solution $U \in W(0, 1)$ we have that

$$(6.2) \quad \rho_0 = \sup_{[0,1]} |u(t)|_{V_\delta} < \infty.$$

Denote the solution of the equation (4.23) by R for an appropriate $\Delta T_1 \leq 1$. Since the constant C in this formula depends continuously on ω and $(\omega \otimes_S \omega)$ we have $|||U_n||| \leq 2R$ if n is chosen sufficiently large. In addition, we choose a ΔT less than or equal to ΔT_1 such that with $u_0^1 = u_0^2$ we have that

$$C\Delta T^{\beta' - \beta}(1 + \Delta T^{2\beta}4R^2) < \frac{1}{2}.$$

Using the notation we have introduced in front of Lemma 4.12 we have

$$\begin{aligned} |||U_n - U||| &\leq |||\mathcal{T}(U, \omega, (\omega \otimes_S \omega), u_0) - \mathcal{T}(U_n, \omega, (\omega \otimes_S \omega), u_0)||| \\ &\quad + |||\mathcal{T}(U_n, \omega, (\omega \otimes_S \omega), u_0) - \mathcal{T}(U_n, \omega^n, (\omega^n \otimes_S \omega^n), u_0)|||. \end{aligned}$$

Hence for a given $\epsilon > 0$ by Lemmata 4.10 and 4.12 we have $|||U_n - U||| < \epsilon$ on $W(0, \Delta T)$. Now we consider $[\Delta T, 2\Delta T \wedge 1]$. We choose n sufficiently large such that $|u(\Delta T) - u_n(\Delta T)|_{V_\delta}$ is sufficiently small (see (4.25)) such that we can repeat the same construction on $W(\Delta T, 2\Delta T \wedge 1)$. In particular, $(U_n)_{n \in \mathbb{N}}$ is a Cauchy sequence on this *Banach space* by Lemmata 4.10 and 4.12, and Remark 4.11. The Chen equality ensures that $(U_n)_{n \in \mathbb{N}}$ is a Cauchy sequence on $W(0, 2\Delta T \wedge 1)$ consisting of solutions of (4.2), (4.3). Due to Lemmata 4.10 and 4.12 we obtain that $(U_n)_{n \in \mathbb{N}}$ converges to the solution U on $W(0, 2\Delta T \wedge 1)$ and in particular $|||U_n - U||| < \epsilon$ on $W(0, 2\Delta T \wedge 1)$ for sufficiently large n . We can cover $[0, 1]$ by *finitely many* intervals of length ΔT . Hence repeating the above construction finitely often we obtain that $|||U_n - U||| < \epsilon$ on $W(0, 1)$ for $n > n_0(\epsilon)$. \square

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